Commutative Algebra Seminar: Lecture of January 30, 2003

The following extremely easy fact makes the notion of phantom homology very useful.

Proposition. *Let $R \to S$ be a homomorphism of Noetherian rings of positive prime characteristic p. Suppose that persistence of tight closure holds for this homomorphism, and suppose that S is weakly F-regular. Let $\cdots \to A_{i+1} \to A_i \to A_{i-1} \to \cdots$ have phantom homology at A. Then the induced map of homology $H_i(A_\bullet) \to H_i(S \otimes_R A_\bullet)$ is zero.*

Proof. Let z be a cycle in A_i, and let B be the module of boundaries, which is the image of A_{i+1}. In the complex $S \otimes_R A_\bullet$, the new module of boundaries is the span of the image of B, and so by the persistence of tight closure, z is in the tight closure of the boundaries in $S \otimes_i A_i$. Since we have weak F-regularity over S, every submodule of every finitely generated module is tightly closed, and so the image of z is a boundary in $S \otimes_R A_i$. $\square$

We now can state and prove a strengthened version of the vanishing theorem for maps of Tor. We gave a different proof earlier for the case where S is regular.

Theorem (vanishing theorem for maps of Tor, second version). *Let $A \to R \to S$ be homomorphisms of Noetherian rings of characteristic $p > 0$ such that A is a regular domain, R is module-finite and torsion-free over A, and $\widehat{S_{\mathcal{M}}}$ is weakly F-regular for every maximal ideal $\mathcal{M}$ of S (which holds if S is weakly F-regular and locally excellent or if S is regular). Then for every A-module M, the map $Tor_i^A(M, R) \to Tor_i^A(M, S)$ is zero.*

Proof. Since Tor commutes with direct limits, we can reduce to the case where M is a finitely generated A-module. Second, if we have a nonzero element in the image of $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$, we may preserve this by localizing at maximal ideal of S that contains the annihilator of that element, and we may replace S by its completion. We may also replace A by $A' = \widehat{A_Q}$ where Q is the contraction of the maximal ideal of S (and R by $R \otimes_A A'$). Thus, we may assume that A is complete local regular. Let $G_\bullet$ be a finite free resolution of M by finitely generated free modules. It satisfies the acyclicity criterion. But heights are preserved when we pass from A to R. Thus, $R \otimes_A G_\bullet$ satisfies the phantom acyclicity criterion, and has phantom homology. The Proposition immediately yields the desired conclusion. $\square$

We have already observed that if R is regular local and excellent then R^+ is faithfully flat over R. We want to note that if R is a polynomial ring over a field, then $R^{+\mathrm{gr}}$ is free as an R-module. In fact, if M is an $\mathbb{N}$-graded module over the polynomial ring $K[x_1, \ldots, x_d]$ such that $x_1, \ldots, x_d$ is a regular sequence on M, then M is free. (Choose homogeneous elements in M whose images in $M/(x_1, \ldots, x_n)M$ give a K-basis. We prove by induction on d that these elements are a free basis, which is clear if $d = 0$. They span M by the homogeneous version of Nakayama's lemma. If $d \geq 1$ and there is a nonzero relation, then there is a homogeneous nonzero relation with a coefficient of smallest degree. By the induction hypothesis the images of the elements form a free basis for M/x_1M over $K[x_2, \ldots, x_d]$. It follows that x_1 divides all coefficients of the relation. Since x_1 is not a zerodivisor on M, we may factor it out to get a relation with a coefficient of lower degree, a contradiction.)

We next give a very simple proof of Faltings' connectedness theorem in equal characteristic p. Cf. [G. Faltings, *A contribution to the theory of formal meromorphic functions*, Nagoya Math. Journal **77** (1980) 99–106], [G. Faltings, *Some theorems about formal functions*, Publ. R.I.M.S. Kyoto **16** (1980) 721–737], and [M. Brodmann and J. Rung, *Local cohomology and the connectedness dimension in algebraic varieties*, Comm. Math. Helv. **61** (1986) 481–490], specifically, Theorem 6 on p. 734 of the second listed paper of Faltings. The paper of Brodmann and Rung obtains the mixed characteristic case.

We want to emphasize the very simple nature of the proof, given the Cohen-Macaulay property for R^+.

Theorem (Faltings). *Let (R, m, K) be a complete local domain of positive prime*[1] *characteristic p. Suppose that R has Krull dimension d and that $I = (x_1, \dots, x_k)R$ is an ideal of R generated by $k \le d-2$ elements. Then the punctured spectrum of R/I is connected.*

Proof. First note that disconnecting the punctured spectrum of R/I is equivalent to giving ideals J and J' of R such that
(1) I is contained in the radicals of J and J'
(2) $JJ' \subseteq I$ (if one only has that $J \cap J' \subseteq I$ one may replace J and J' by powers such that the product is contained in I).
(3) $J + J'$ is primary to m.
(4) Neither J nor J' is primary to m.

We shall first do the proof in the case where $x_1, \dots, x_k$ is part of a system of parameters for R: then we shall show that the general case reduces to this one. Since $J + J'$ is primary to m and $\dim R \ge k+2$, we can choose $y+y'$, $z+z'$ where y, $z \in J$ and y', $z' \in J'$ such that $x_1, \dots, x_k$, $y+y'$, $z+z'$ is part of a system of parameters for R. Modulo $I = (x_1, \dots, x_k)R$ one has the relation $z(y+y') - y(z+z') = 0$, because zy', $yz' \in JJ' \subseteq I$. Since parameters form a regular sequence in R^+, we have that $y \in (y+y')R^+ \bmod IR^+$, say $y \equiv u(y+y')$ mod IR^+. Similarly, $y' \equiv v(y+y')$ mod IR^+. Then $(1-u-v)(y+y') = 0$ in R^+/IR^+, while $y+y'$ is not a zerodivisor in R^+/IR^+. It follows that $1-u-v \in IR^+$. Therefore at least one of the elements u, v must be a unit in R^+. Suppose that it is u: the argument for the case where it is v is entirely similar. Then $y \in J$ is not a zerodivisor in R^+/IR^+, and since $JJ' \subseteq I$, we have that $yJ'R^+ \subseteq IR^+$ and so $J'R^+ \subseteq IR^+$. It follows that $(J+J')R^+ \subseteq (\operatorname{Rad} J)R^+$. But a minimal prime $Q \ne m$ of J will have a prime $\mathcal{Q}$ of R^+ lying over it, and then $J + J' \subseteq \mathcal{Q}$. This implies that $J + J' \subseteq Q$, contradicting the fact that $J + J'$ is primary to m.

[1] On p. 79 of [M. Hochster and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992) 53–89] it is explained how to deduce the equal characteristic 0 case from the characteristic $p > 0$ case.