Commutative Algebra Seminar: Lecture of February 6, 2003

Before giving the rest of the argument, we pause to prove the following fact.

Lemma. *Suppose that (R, m, K) is an equidimensional catenary local ring. Suppose also that $x_1, \ldots, x_k \in m$ are contained in an ideal $\mathfrak{A} \subseteq m$ of height at least k. Then there exist $y_1, \ldots, y_k \in \mathfrak{A}$ that are part of a system of parameters such that $(x_1, \ldots, x_k)R$ is contained in the radical of $(y_1, \ldots, y_k)R$.*

Proof. The general case follows easily once we do the case where $k = 1$. Suppose $x = x_1$ is given. If x is part of a system of parameters, simply take $y = x$. If x is nilpotent we may take y to be any element of $\mathfrak{A}$ that is not in any minimal prime (possible by prime avoidance, since $\mathfrak{A}$ has height at least 1). In the general case, assume that x is in certain minimal primes P_i indexed by i and not in certain other minimal primes P_j indexed by j. Choose $y_0 \in \mathfrak{A}$ not in any minimal prime. Choose $z_j \in P_j$ not in any other minimal prime, and let z be the product of the z_j. Then $zy_0 \in \mathfrak{A}$ and is precisely in those minimal primes to which x does not belong. Then $y = x + zy_0$ is not in any minimal prime, and so is part of a system of parameters, while $\operatorname{Rad} xR = \operatorname{Rad} x^2R = \operatorname{Rad} x(x + zy_0)R$ (since xz is nilpotent, so that xzy_0 is nilpotent as well) $\subseteq \operatorname{Rad}(x + zy_0)R$.

We complete the argument by induction on k. Suppose that $y_1, \ldots, y_{k-1} \in \mathfrak{A}$ have been chosen so that these elements are part of a system of parameters, and such that $(x_1, \ldots, x_{k-1})R \subseteq \operatorname{Rad}(y_1, \ldots, y_{k-1})R$. The fact that we can choose y_k now follows from the case where $k = 1$: replace R by $R/(y_1, \ldots, y_{k-1})R$, the x's by the image of x_k, and $\mathfrak{A}$ by its image in $R/(y_1, \ldots, y_{k-1})R$, which will have height at least 1. Because R is equidimensional and catenary of dimension d, $R/(y_1, \ldots, y_{k-1})R$ is equidimensional and catenary of dimension $d - (k - 1)$, i.e., the quotient of R by any minimal prime of $(y_1, \ldots, y_{k-1})R$ has dimension $d - (k - 1)$. $\square$

Proof of Faltings' theorem, continued. We can now use the Lemma above to reduce to the case where I is generated by part of a system of parameters. With J and J' as before choose $\mathfrak{B}$, $\mathfrak{B}'$ to be any ideals of R of height at least k, not primary to m, such $\operatorname{Rad} J \subseteq \mathfrak{B}$, $\operatorname{Rad} J' \subseteq \mathfrak{B}'$, and let $\mathfrak{A} = \mathfrak{B} \cap \mathfrak{B}'$. Choose a sequence of parameters $y_1, \ldots, y_k \in \mathfrak{A}$ such that $x_1, \ldots, x_k$ are in $\operatorname{Rad}(y_1, \ldots, y_k)R$, using the Lemma. But then the ideals $(y_1, \ldots, y_k)R + J$ and $(y_1, \ldots, y_k)R + J'$ disconnect the punctured spectrum of $R/(y_1, \ldots, y_k)R$ (note that the product is contained in the radical of $(y_1, \ldots, y_k)$, since $JJ' \subseteq (x_1, \ldots, x_k) \subseteq \operatorname{Rad}(y_1, \ldots, y_k)$). This is impossible since we have already proved the result for sequences of parameters. $\square$

A Noetherian domain that is a direct summand, as a module over itself, of every module-finite extension domain is called a *splinter*. For rings containing the rational numbers, this condition simply characterizes the normal (integrally closed) domains, as we shall after the proof of the Theorem below. In characteristic p it was proved last semester (see the initial Corollary in the notes on the lecture of September 26) that a weakly F-regular domain is a splinter. It is conjectured that a locally excellent Noetherian domain of positive prime characteristic p is a splinter if and only if it is weakly F-regular. This is known in the Gorenstein case and slightly more generally. We shall discuss the situation again after we have proved Karen Smith's result that plus closure agrees with tight closure for parameter ideals in the case of a locally excellent domain. The property of being a splinter is easily shown to pass to localizations[1] and so the conjecture, if it could be proved, would settle the very important conjecture that weakly F-regular implies F-regular in the locally excellent case.

While we cannot prove that characteristic p splinters are weakly F-regular, we can prove that they are Cohen-Macaulay, as an easy consequence of the Cohen-Macaulay property of R^+ (this is quite false for rings containing $\mathbb{Q}$: we shall give an explicit example of a normal domain finitely generated over a field of characteristic 0 of dimension 3 that is not Cohen-Macaulay below):

Theorem. *Let R be a locally excellent domain that is a direct summand of every module-finite extension domain. Then R is Cohen-Macaulay.*

Proof. The issue is local on R, and, as noted above, for every prime P, R_P also splits from every module-finite extension domain. We may therefore assume that R is excellent local, so that R^+ is a big Cohen-Macaulay algebra. Now suppose that $x_{k+1}r_{k+1} = \sum_{i=1}^k r_i x_i$ for part of a system of parameters $x_1, \ldots, x_{k+1}$ in R, with the r_i in R. Then $r_{k+1} = \sum_{i=1}^k s_i x_i$ with the $s_i \in R^+$, and we may let $T = R[s_i : i]$. Applying an R-module retraction $\phi : T \to R$, we have that $r_{k+1} = \sum_{i=1}^k \phi(s_i)x_i \in (x_1, \ldots, x_k)R$, as required. $\square$

We now come back to the problem of understanding splinters in equal characteristic zero. Suppose that $\mathbb{Q} \subseteq R \subseteq S$ and that S is module-finite over R. Let $\mathcal{K} \subseteq \mathcal{L}$ be the fraction fields of R and S respectively, and let $d = [\mathcal{L} : \mathcal{K}]$. Let $\mathrm{tr} = \mathrm{tr}_{\mathcal{L}/\mathcal{K}}$ denote field trace. If $v \in \mathcal{L}$, multiplication by v is a $\mathcal{K}$-linear map of the d-dimensional vector space $\mathcal{L}$ to itself. A matrix for this map has a trace which is independent of the choice of basis for $\mathcal{L}$ over $\mathcal{K}$, since similar matrices have the same trace, and this is $\mathrm{tr}\,(v)$. Then $\mathrm{tr} : \mathcal{L} \to \mathcal{K}$ is $\mathcal{K}$-linear, and is simply multiplication by d on $\mathcal{K}$, since the matrix of an element $u \in \mathcal{K}$ is diagonal with d copies of u on the diagonal.

We next observe that if R is normal, then the restriction of tr to S has values in R. One way to see this is to note that if R is normal, it is an intersection of the discrete valuation rings $V = R_P$ as P runs through the height one primes of R.[2] Note that we are also using here that a one-dimensional Noetherian normal local domain is a DVR. It therefore suffices to show that $\mathrm{tr}\,(S_P) \subseteq R_P$ for height one primes P of R, and so we have reduce to the case where $R = V$ is a DVR. But then S is free over R (since it is finitely generated and torsion-free, while R is a principal ideal domain), and a free basis for S over

[1] Every module finite extension of $S^{-1}R$, where R is a domain and S is a multiplicative system, arises as a localization of a module-finite extension of R, for if u satisfies a monic polynomial over $S^{-1}R$, with $s \in S$ as a common denominator for the coefficients other than the first, then su is integral over R.

[2] If $a/b \in \mathcal{K} - R$ with a, $b \neq 0$, then the associated primes of b are all height one, since R is normal: we can choose a height one associated prime P of bR such that a is not in the P-primary component of bR, and then $a \notin bR_P$, i.e., $a/b \notin R_P$.

R will also be a basis for $\mathcal{L}$ over $\mathcal{K}$. But when we compute $\mathrm{tr}\,(v)$ for $v \in S$ using that free basis, all entries of the matrix are in S, and so its trace is in S, as required.

Recall that if we have $R \to S$ with R a direct summand of S, then $IS \cap R = I$ for every ideal I of S: if $\phi : S \to R$ is an R-linear retraction and $r = \sum_{j=1}^{k} s_j f_j$ with $r \in R$, $f_j \in I$ and s_j in S, we may apply ϕ to get $r = \phi(r) = \sum_{j=1}^{k} \phi(s_j) f_j$, since the $f_j \in R$. But every $\phi(s_j) \in R$, and so $r \in I$.

This means that if R is a normal local ring containing $\mathbb{Q}$ and we have a non-trivial relation on parameters, say $rf_{k+1} \in (f_1, \dots, f_k)R$ with $r \notin (f_1, \dots, f_k)S$, we cannot get this relation to become trivial by passing to R^+. For if $r \in (f_1, \dots, f_k)R^+$, then $r \in (f_1, \dots, f_k)S$ for S module-finite over R. But then, by the paragraph above, $r \in (f_1, \dots, f_k)R$, a contradiction. Thus, if one starts with a normal local non-Cohen-Macaulay ring R containing $\mathbb{Q}$, R^+ cannot be a big Cohen-Macaulay algebra for R.

We want to give a completely specific example in dimension 3 of a normal ring that is not Cohen-Macaulay. First recall that if B and C are $\mathbb{N}$-graded K algebras with $B_0 = C_0 = K$, then the *Segre product* of B and C over K, which we denote $B \underset{\mathrm{seg}}{\widetilde{\otimes}}_K C$, is

$$\bigoplus_{j=1}^{\infty} B_j \otimes_K C_j \subseteq B \otimes_K C.$$

If B_1 generates B over K and C_1 generates C over K, then $B_1 \otimes_K C_1$ generates $B \underset{\mathrm{seg}}{\widetilde{\otimes}}_K C$ over K. (Note also that $\mathrm{Proj}\,(B \underset{\mathrm{seg}}{\widetilde{\otimes}}_K C) \cong \mathrm{Proj}\,B \times \mathrm{Proj}\,C$: the Segre product of rings corresponds to the Segre embedding of the product of two projective varieties as a closed set in projective space.) $B \underset{\mathrm{seg}}{\widetilde{\otimes}}_K C$ is a direct summand of $B \otimes_K C$ as a module over itself: as a complement, one may take $\bigoplus_{i \neq j} B_i \otimes_K C_j$. We want to point out as well that a direct summand R of a normal domain T is normal.[3]

Now let $B = K[X,\,Y,\,Z]/(X^3 + Y^3 + Z^3) = K[x,\,y,\,z]$ where K is any field of characteristic $\neq 3$. Let $C = K[s,t]$, a polynomial ring. B is a surface with an isolated singularity and is normal. Then $R = B \underset{\mathrm{seg}}{\widetilde{\otimes}}_K C = K[xs,\,ys,\,zs,\,xt,\,yt,\,zt] \subseteq B[s,\,t]$ is normal, for it is a direct summand of $B[s,\,t]$, which is normal because B is. Note that R is module-finite over $D = K[xs,\,ys,\,xt,\,yt]$, since $(zs)^3 = -(xs)^3 - (ys)^3$ and $(zt)^3 = -(xt)^3 - (yt)^3$. $D \cong K[a,b,c,d]/(ad - bc)$ and so D and R are three-dimensional. It is easy to see that $a,\,d,\,b - c$ is a homogeneous system of parameters in D, and so $xs,\,yt,\,ys - xt$ is a homogeneous system of parameters in R as well. But R is *not* Cohen-Macaulay, because of the non-trivial relation $\big((zs)(zt)\big)(ys - xt) = (zs)^2 yt - (zt)^2 xs$. One needs to see that $(zs)(zt) \notin (yt,\,xs)R$. But this is true even when we replace R by the larger ring $B[s,t]$, where it is easy to see because we can specialize both s and t to 1, and we still have $z^2 \notin (y,\,x)B$. Thus, R is three-dimensional, normal and not Cohen-Macauly, and all this is preserved when we localize at the homogeneous maximial ideal m. It follows that $(R_m)^+$ is not a big Cohen-Macaulay algebra over R_m when K is chosen to be a field of characteristic 0.

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[3] If a/b, a, $b \in R - \{0\}$ is integral over R, it is integral over T, and so in T, yielding that $a \in bT \cap R = bR$, i.e., that $a/b \in R$.