Commutative Algebra Seminar: Lecture of February 13, 2003

We can also see that T^+ cannot be a big Cohen-Macaulay algebra in a good sense when T is excellent, local of dimension 4, and mixed characteristic $p > 0$, with $p \neq 3$. (There are similar examples in mixed characteristic 3.) Let V be the p-adic numbers and form $B = V[X, Y, Z]/(X^3 + Y^3 + Z^3) = V[x, y, z]$, for example. Let $T = V[sx, sy, sz, tx, ty, tz] \subseteq B[s, t]$. Localize at the maximal ideal generated by p and the elements adjoined. Again, sx, ty, $sy - tx$ is part of a system of parameters (together with p, these elements form a full system of parameters). The non-trivial relation on these that we considered earlier is still present here and cannot be trivialized when we pass to T^+: if that happened, then it would also be trivial in $(T_Q)^+$, where Q is the prime ideal generated by sx, sy, sz, tx, ty, tz. But now we are in the case already handled of dimension 3 over a ring containing the rationals.

So far as I know, it is an open question whether R^+ is a big Cohen-Macaulay algebra for excellent mixed characteristic local domains R of dimension 3. Ray Heitmann has recently proved a weaker result, using an extraordinarily difficult equational "forcing" lemma, for such rings: every Koszul relation becomes trivial in R^+ for such a ring after it is multiplied by $p^{1/N}$ for any choice of positive integer N. Cf. [R. Heitmann, *The direct summand conjecture in dimension three* Annals of Math. (2) **156** (2002) 695–712]. Heitmann used this to prove that regular rings are direct summands of their module-finite extension rings in dimension 3. (The same question is open in mixed characteristic in dimension ≥ 4.) Heitmann's result can be used to prove the existence of big Cohen-Macaulay algebras in dimension three: see [M. Hochster, *Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem*, J. of Algebra **254** (2002) 395–408]. Essentially the same argument given earlier shows that Heitmann's result cannot hold in dimension 4 and higher however: all the elements $p^{1/N}$ become units when one localizes at Q.

We next want to give a characterization of when an element is in the tight closure of an ideal that is somewhat different in nature from any of our earlier characterizations. In order to do this we need some local cohomology theory. We shall also need local cohomology theory for the proof that tight closure is the same as plus closure for parameter ideals in locally excellent domains. We therefore pause to give a summary of some relevant results.

Let R be a Noetherian ring, let I be an ideal of R, and let $f_1, \ldots, f_n$ be generators of I. Let M be an R-module, which need not be finitely generated. We shall give three different descriptions of how to calculate the local cohomology modules $H^i_I(M)$. However, we first note that there is a very simple description if $i = 0$. In this case, we let $H^0_I(M) = \bigcup_t \mathrm{Ann}_M I^t$. Notice that the term indexed by t in this union is the same as $\mathrm{Hom}_R(R/I^t, M)$. Now, here are the three methods for extending this definition of local cohomology modules to higher dimension.

In this paragraph only, R need not be Noetherian. We review some facts about injective modules and essential extensions. E is *injective* if every map from a submodule $N \subseteq M$ to E extends to all of M, or, equivalently, $\mathrm{Hom}_R(\_\_, E)$ is an exact functor. Note that a module E is injective if and only if[1] every map from an ideal $I \subseteq R$ to E extends to R. An extension Q of an R-module M is called *essential* if every nonzero submodule of Q has nonzero intersection with M. It is equivalent to say that every nonzero element of Q has a nonzero multiple in M, or that every map from Q that is injective when restricted to M is injective. By a theorem, every R-module can be embedded in an injective R-module.[2] An R-module has a *maximal essential extension*, unique up to non-unique isomorphism, and this maximal essential extension is injective, by another theorem. It is called the *injective hull* of M over R, and denoted $\mathcal{E}_R(M)$. If $M \subseteq E$ where E is any injective module, then one can show there will be an isomorphic copy of $\mathcal{E}_R(M)$ between M and E, and it will be a direct summand of E. Every R-module has injective resolution. One can embed M in an injective E_0, and then embed E_0/M in an injective E_1, etc. If one uses an injective hull at every stage, the resolution is called *a minimal injective resolution.* Minimal injective resolutions are unique up to non-unique isomorphism of complexes.

There is a natural map $\mathrm{Ext}^i_R(R/I^t,\, M) \to \mathrm{Ext}^i_R(R/I^{t+1},\, M)$ induced by the obvious surjection $R/I^{t+1} \twoheadrightarrow R/I^t$. We may define $H^i_I(M)$ as the direct limit. Note that this definition agrees with the definition already given for $H^0_I(M)$. These are covariant functors of M. They vanish when M is injective. There is a functorial long exact sequence for this cohomology, which is obtained by taking the direct limit of long exact sequences for $\mathrm{Ext}^i_R(R/I^t,\, \_)$. Thus, when $0 \to M' \to M \to M'' \to 0$ is exact there is a long exact sequence:

$$0 \to H^0_I(M') \to H^0_I(M) \to H^0_I(M'') \to \cdots$$

$$\to H^i_I(M') \to H^i_I(M) \to H^i_I(M'') \to H^{i+1}_I(M') \to \cdots$$

Every element of every $H^i_I(M)$ is killed by a power of I. It is easy to see that if J has the same radical as I then the powers of J and the powers of I are cofinal, and it follows that $H^i_J(M) \cong H^i_J(M)$ for all i.

A second method is choose an injective resolution of M, say $M \to E_0 \to E_1 \to \cdots$, and then form the cohomology of the complex $0 \to H^0_I(E_0) \to H^0_I(E_1) \to \cdots$. It is not difficult to prove that this agrees with the direct limit definition given earlier, if one uses the same injective resolution to calculate $\mathrm{Ext}^\bullet_R(R/I^t,\, M)$ for every t. Taking direct limits of complexes commutes with calculation of cohomology.

[1]Given $N \subseteq M$ and $f : N \to M$, use Zorn's lemma to extend f maximally to a map g to E on $N' \supseteq N$. If $N' \neq M$, choose $u \in M - N'$. It suffices to extend g to $N' + Ru$. Let $I = \{i \in R : iu \in N'\}$. The map $h : I \to E$ given by $h(i) = g(iu)$ extends to h' on R: let $e = h'(1)$. Then $g(iu) = ie$ for all $i \in I$. g extends to $N' + Ru$ by letting its value on $v + ru$ be $g(v) + re$. This is well-defined, because if $v_1 + r_1u = v_2 + r_2u$ then $v_2 - v_1 = (r_1 - r_2)u$, so that $r_1 - r_2 = i \in I$, and $g(v_2) + r_2e = g(v_1) + g(v_2 - v_1) + (r_1 - i)e = g(v_1) + g((r_1 - r_2)u)) + r_1e - ie = g(v_1) + r_1e + g(iu) - ie = g(v_1) + r_1e$.

[2]By the result mentioned above, an R-module E is injective iff maps $I \to E$, $I \subseteq R$, extend to R. If R is a PID, this is easily seen to mean that E is *divisible*, i.e., that for all $r \in R - \{0\}$, $rE = E$. It follows that $W = \mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, and one can check that if $N^\vee$ denotes $\mathrm{Hom}_\mathbb{Z}(N,\, W)$, then for every $\mathbb{Z}$-module N, the natural map $N \to N^{\vee\vee}$ is injective. Given any R-module M, map a free R-module $G \twoheadrightarrow M^\vee$. Then we have $M \hookrightarrow M^{\vee\vee} \hookrightarrow G^\vee$. But $G^\vee$ is injective as an R-module! The functor $\mathrm{Hom}_R(\_,\, G^\vee)$ may be identified with the functor $(\_ \otimes_R G)^\vee$ by the adjointness of tensor and Hom, and the latter is exact because G is R-free (and so R-flat) and $^\vee$ is exact.

A very different and very useful alternative is the following. Let $I = (f_1, \dots, f_n)$. Let $\underline{f}$ denote $f_1, \dots, f_n$. We shall write $\mathcal{K}^\bullet(\underline{f}^\infty; R)$ for the tensor product of the n complexes $0 \to R \to R_{f_j} \to 0$, which looks like this:

$$0 \to R \to \bigoplus_j R_{f_j} \to \bigoplus_{j_1, j_2} R_{f_{j_1} f_{j_2}} \to \cdots \to R_{f_1 \cdots f_n} \to 0.$$

The i th term is the direct sum of the localization $R_{f_{t_1} \cdots f_{t_i}}$ as $\{t_1, \dots, t_i\}$ runs through all i element subsets of $\{1, \dots, n\}$. The maps have signs similar to those used in Koszul complexes, but are otherwise induced by the natural maps $R_{f_1 \cdots f_i} \to R_{f_1 \cdots f_i f_{i+1}}$ that come from the fact that the target is the localization of the domain at the image of f_{i+1}. Clearly, $\mathcal{K}^\bullet(\underline{f}^\infty; R)$ is a complex of flat R-modules. Let $\mathcal{K}^\bullet(\underline{f}^\infty;\, M) = \mathcal{K}^\bullet(\underline{f}^\infty;\, R) \otimes_R M$, which is the complex:

$$0 \to M \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1, j_2} M_{f_{j_1} f_{j_2}} \to \cdots \to M_{f_1 \cdots f_n} \to 0.$$

We now define (temporarily) $\mathcal{H}^i_{\underline{f}}(M)$ as $H^i\big(\mathcal{K}^\bullet(\underline{f}^\infty;\, M)\big)$. We claim that $\mathcal{H}^i_{\underline{f}}(M) \cong H^i_I(M)$, which we have defined earlier, as functors of M. Thus, it does not matter how we choose the generators for I. In fact, since we have already observed that $H^i_I(M)$ only depends on what I is up to radicals, we only need to choose the f_j so that the radical of the ideal they generate is the same as the radical of I.

One point worth making is that $\mathcal{H}^0_{\underline{f}}(M)$ is, by definition, the kernel of the map $M \to \bigoplus_j M_{f_j}$, and we leave it to the reader to verify that this is precisely the submodule of M consisting of all elements killed by a power of I. Second, given a short exact sequence $0 \to M' \to M \to M'' \to 0$ one may tensor with the flat complex $\mathcal{K}^\bullet(\underline{f}^\infty;\, R)$ to get a short exact sequence of complexes. The corresponding long exact sequence of cohomology shows that our new theory has a functorial long exact sequence. Thus, we already know:

(1) Both $\mathcal{H}^0_{\underline{f}}(M)$ and $H^0_I(M)$ are $\bigcup_t \operatorname{Ann}_M I^t$.

(2) The covariant functors $\mathcal{H}^i_{\underline{f}}$ and H^i_I both have functorial long exact sequences.

We also know that

(3) $H^i_I(M) = 0$ for $i > 0$ if M is injective.

To show that $\mathcal{H}^i_{\underline{f}}(M) \cong H^i_I(M)$ canonically for all M, we need to verify that $\mathcal{H}^I_{\underline{f}}(M)$ vanishes for $i > 0$ when M is injective (this will take some work), and then show that the fact that the two cohomology theories share properties (1), (2), and (3) implies that they are the same. Before starting on the remaining steps, we note, in summary, two more properties that we have established for H^i_I:

(4) Every element of $H^i_I(M)$ is killed by a power of I.

(5) If $\operatorname{Rad} I = \operatorname{Rad} J$, then $H^i_I(M) \cong H^i_J(M)$.