Commutative Algebra Seminar: Lecture of February 20, 2003

We want to look at local cohomology in yet another way. First observe that the complex $0 \to R \to R_{f_j} \to 0$ is the direct limit over t of the complexes $0 \to R \xrightarrow{f_j^t\,\cdot} R \to 0$, where we use the maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{f_j^{t+1}\,\cdot} & R & \longrightarrow & 0 \\ & & =\big\uparrow & & \big\uparrow f_j\,\cdot & & \\ 0 & \longrightarrow & R & \xrightarrow{f_j^{t}\,\cdot} & R & \longrightarrow & 0 \end{array}$$

One needs to see that the direct limit system in which each module is R and each map is multiplication by f has direct limit R_f. (Index the copies of R by $\mathbb{N}$ and map the generator 1 in the tth copy to $1/f^t$ in R_f. Note that the ideal J of all elements killed by a power of f in R maps to 0 in the direct limit, no matter which copy of R we consider. Let $\overline{R} = R/J$ and $\overline{f}$ be the image of f. Then the problem is unaffected if we replace R by $\overline{R}$ and f by $\overline{f}$. The remaining details are left to the reader.) We write $\mathcal{K}^\bullet(\underline{f};\, R)$ for the result of tensoring together the n complexes $0 \to R \xrightarrow{f_j\,\cdot} R \to 0$, each numbered so that the leftmost copy of R is in degree 0 and the rightmost copy of R in degree 1. (This is the same, up to isomorphism, as the homological Koszul complex introduced earlier, except that it is numbered "backwards.") Likewise we write $\mathcal{K}^\bullet(\underline{f};\, M)$ for $\mathcal{K}^\bullet(\underline{f};\, R) \otimes_R M$.

Then we see that $\mathcal{K}^\bullet(\underline{f}^\infty;\, R)$ is the direct limit of the "cohomological" Koszul complexes $\mathcal{K}^\bullet(f_1^t,\, \ldots\,,\, f_n^t; R)$ obtained by tensoring together the complexes $0 \to R \xrightarrow{f_j^t\,\cdot} R \to 0$, and likewise, $\mathcal{K}^\bullet(\underline{f}^\infty;\, M)$ is the direct limit of the $\mathcal{K}^\bullet(f_1^t,\, \ldots\,,\, f_n^t; M)$. Since taking direct limits of complexes commutes with taking cohomology, we also find that $\mathcal{H}^i_{\underline{f}}(M)$ is the direct limit in t of $H^i(f_1^t,\, \ldots\,,\, f_n^t;\, M) \cong H_{n-i}(f_1^t,\, \ldots\,,\, f_n^t; M)$. Thus, every element of $\mathcal{H}^i_{\underline{f}}(M)$ is killed by a power of $I = (\underline{f})R$, as we would expect, since we aim to show that it is the same as $H^i_I(M)$.

To see why the two cohomology theories will agree if we show that $\mathcal{H}^i_{\underline{f}}(E) = 0$ when E is injective and $i \geq 1$, note that they already agree if $i = 0$, and that given any module M we can form a short exact sequence $0 \to M \to E \to C \to 0$ which yields a long exact sequence for each cohomology theory. Because the higher cohomology of E vanishes for both theories, if $H^i(\_)$ denotes either theory the long exact sequence yields an exact sequence:

$$0 \to H^0(M) \to H^0(E) \to H^0(C) \to H^1(M) \to 0$$

for each cohomology theory, and since the first three terms are the same for the two theories (including the maps) we obtain an isomorphism $H^1_I(M) \to \mathcal{H}^1_{\underline{f}}(M)$ for all M. It is not hard to check that it is independent of the choice of embedding, and that one similarly gets an isomorphism of functors. (To deal with maps, note that given a map $M \to M'$ and short exact sequences $0 \to M \to E \to C \to 0$ and $0 \to M' \to E' \to C' \to 0$, the composite map $M \to E'$ extends to all of E, and the map $E \to E'$ one gets induces a map $C \to C'$. Thus, one gets a map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & E & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & E' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

Now make use of the functoriality of the long exact sequences for the two functors. Again, one must check independence of choices made.) At the same time, the long exact sequence yields isomorphisms $H^i(C) \to H^{i+1}(M)$ for all i for both choices of H^i. It follows by induction on i that $H^i_I(M) \cong \mathcal{H}^i_{\underline{f}}(M)$ for all i, and it is not hard to check, likewise, that one has an isomorphism of functors for all i.

Before we actually prove that $\mathcal{H}^i_{\underline{f}}(E) = 0$ for injective modules E over Noetherian rings and for all $i \geq 1$, we need to make one more observation about the behavior of the $\mathcal{H}^i_{\underline{f}}(M)$ under change of rings, and we also need to understand something about the structure of injective modules over Noetherian rings.

First note that if $R \to S$ is a homomorphism of Noetherian rings and M is an S-module, then if $f_1, \ldots, f_n \in R$ have images $g_1, \ldots, g_n$ in S, then $\mathcal{H}^i_{\underline{f}}(M) = \mathcal{H}^i_{\underline{g}}(M)$ as S-modules: on the left, M is considered as an R-module via restriction of scalars. The reason is that the localization of M at a product of the f_j is the same as its localization at the product of the corresponding g_j, so that the two complexes involved whose cohomology we are taking are actually the same.

Once we have finished showing that the two cohomology theories are the same, it will follow that $H^i_I(M) \cong H^i_{IS}(M)$ whenever M is an S-module! This is very important, but we cannot use it until we have finished verifying that $\mathcal{H}^i_{\underline{f}}(E) = 0$ for injective modules E when $i \geq 1$.

We next observe that over a Noetherian ring R, every direct sum of injective modules is injective. (In general, any product, even an infinite one, of injectives is injective over any ring, but this is not true for infinite direct sums. Finite direct sums of injectives are injective because they are the same as finite products.) In the Noetherian case, to prove that a possibly infinite direct sum is injective, it suffices to show that a map from an ideal of R to the infinite direct sum extends to all of R. Fix a finite set of generators of the ideal. The image of each generator only involves finitely many of the summands. Thus, the image of the ideal is contained in a finite direct sum of the injective summands, and since this is an injective module, the map can already be extended using just these.

We next observe that every injective module E over the Noetherian ring R is isomorphic to a direct sum of injective hulls $\mathcal{E}_R(R/P)$ of prime cyclic modules R/P. To see this, consider families of submodules of E each of which has the form $\mathcal{E}_R(R/P)$ and such that their sum is direct (i.e., each is disjoint from the sum of the others). An increasing union of such families is again such a family, and so Zorn's lemma implies that there is a maximal such family. Let $E_0 \subseteq E$ be its sum, which is a direct sum. We shall show that $E_0 = E$. Now E_0 is injective, and so the identity map $E_0 \to E_0$ extends to all of E. This means that E_0 is a direct summand of E. We can write $E = E_0 + E_1$ where $E_1 \subseteq E$ and $E_1 \cap E_0 = 0$. We want to see that E_1, which is injective, since it is a direct summand of E, must be 0. If not, choose a nonzero element $u \in E_1$. Let P be an associated prime of Ru. Then $R/P \hookrightarrow Ru \hookrightarrow E_1$. Since E_1 is an injective module and $R/P \subseteq \mathcal{E}_R(R/P)$, this map extends to $\mathcal{E}_R(R/P)$. Since the map is injective on R/P and $R/P \hookrightarrow \mathcal{E}_R(R/P)$ is injective, it follows that the map is injective on $\mathcal{E}_R(R/P)$. But this can now be used to enlarge the family that was used to construct E_0, contradicting the maximality of that family, and we are done.

An essential extension of R/P is not a direct sum in a non-trivial way: if $R/P \hookrightarrow E_1 \oplus E_2$ were essential with both $E_i \neq 0$, then both $E_1 \cap R/P$ and $E_2 \cap R/P$ would be nonzero, and any two nonzero submodules of R/P have nonzero intersection (these are ideals of an integral domain: multiply nonzero elements, one from each, together), contradicting $E_1 \cap E_2 = 0$. Thus, $\mathcal{E}_R(R/P)$ is indecomposable.

Next note that $\mathcal{E}_R(R/P)$ is a module over R/P: if $a \in R - P$, multiplication by a is injective on R/P, and so must be injective on the essential extension $\mathcal{E}_R(R/P)$. But the map must also be onto, or else its image, a submodule of $\mathcal{E}_R(R/P)$ which is $\cong \mathcal{E}_R(R/P)$ and, therefore, injective, would split off, contradicting the preceding paragraph. It follows that it contains a copy of the fraction field of R/P, and is easily seen to be a maximal essential extension of R_P/PR_P. Thus, $\mathcal{E}_R(R/P) \cong \mathcal{E}_{R_P}(R_P/PR_P)$. We therefore go a long way towards understanding all injective modules over Noetherian rings if we understand the injective hull of the residue class field of a local ring.

Let $(R,\, m,\, K)$ be local. Let $\mathcal{E} = \mathcal{E}_R(K)$ be an injective hull for $K = R/m$. We shall say quite a bit more about these modules later. For now, we want to note that every element is killed by a power of m. The reason is that if u is not killed by a power of m, then Ru has an associated prime P other than m, giving an embedding $R/P \hookrightarrow Ru \hookrightarrow \mathcal{E}$, and then this copy of R/P must meet K, a contradiction when $P \neq m$.

Consider any module M over R in which every element is killed by a power of m. Then M is automatically a module over $\widehat{R}$: if $u \in M$, $Ru \cong R/\mathrm{Ann}_R u$, and $\mathrm{Ann}_R u$ is an m-primary ideal. This means that $R/\mathrm{Ann}_R u \cong \widehat{R}/(\mathrm{Ann}_R u)\widehat{R}$, and is thus a module over $\widehat{R}$. Since this is true for every element $u \in M$, we have a unique $\widehat{R}$-module structure M. In consequence, one has that $\widehat{R} \otimes_R M \cong M$. One can also say the following about the way that $\widehat{R}$ operates on M: given $s \in cR$ and u in M we can choose a power m^t of m such that $m^t u = 0$, and we can choose $r \in R$ such that $s \equiv r$ modulo $m^t\widehat{R}$. Then $su = ru$. We also note that for any such module M, if we let the *socle* of M, $\mathrm{Soc}\, M$, be $\mathrm{Ann}_M m$, which is a submodule of M whose structure as an R-module is determined by it structure as a K-vector space, then M is an essential extension of $\mathrm{Soc}\, M$. For if $u \in M - \{0\}$, we can choose a nonnegative integer t such that $m^t u \neq 0$ but $m^{t+1}u = 0$. Then any nonzero element of $m^t u$ is in $\mathrm{Soc}\, M$.

Now, for $(R,\, m,\, K)$ local, we have that $\mathcal{E}_R(K)$ is a module of this sort (and the socle is easily seen to be K). It is also a module over $\widehat{R}$, and it is ease to see that it also a maximal essential extension of K over $\widehat{R}$. This means that $\mathcal{E}_R(K) \cong \mathcal{E}_{\widehat{R}}(K)$. Therefore, to understand injective modules over Noetherian rings, in some sense it suffices to understand injective hulls of residue fields of complete local rings.

We are now more than ready to prove that $\mathcal{H}^i_{\underline{f}}(E) = 0$ when E is injective and $i \geq 1$. The cohomology theory $\mathcal{H}^\bullet_{\underline{f}}(\_)$ commutes with arbitrary direct sums and direct limits. Thus, it suffices to consider the case where $M = \mathcal{E}_R(R/P)$. Since it does not matter which ring we think over in computing $\mathcal{H}^i_{\underline{f}}$, we may assume, without loss of generality, that (R, m, K) is local and that $M = \mathcal{E}_R(R/m)$. Since every element of M is killed by a power of m, it will suffice to show that $\mathcal{H}^i_{\underline{f}}(M) = 0$ for $i \geq 1$, under this hypothesis. Since M is the direct limit of its finitely generated submodules, which will have finite length, and this cohomology theory commutes with direct limits, we need only prove the fact that $\mathcal{H}^i_{\underline{f}}(M) = 0$ for M of finite length and $i \geq 1$. By the long exact sequence for cohomology and induction on the length it suffices to prove that $\mathcal{H}^i \underline{f}(K) = 0$ for $i \geq 1$, where $K = R/m$. We may work over the ring K instead of over R. Thus, we have reduced to considering the case where $R = K = M$. If any f_j has nonzero image in K, we are done because $(f_1, \dots, f_n)K$ is the unit ideal and has a power that kills every element of every $H^i_{\underline{f}}(K)$. Finally, we have reduced to the case where $R = K = M$ and all the f_j are 0. But then every K_{f_j} is 0, and the complex $\mathcal{K}(\underline{f}^\infty;\, K)$ is the tensor product of n complexes, each of which consists simply of K in degree 0 and 0 elsewhere: the tensor product also consists of K in degree 0 and 0 and elsewhere, and we are done.

Thus, we have completed the proof that $H^i_I(M) = \mathcal{H}^i_{\underline{f}}(M)$ as functors for all R-modules M. All of our definitions of local cohomology agree. We want to emphasize that we now know, in consequence, that the following important fact, mentioned earlier, holds:

(6) If $R \to S$ is a homomorphism of Noetherian rings, M is any S-module, and I is an ideal of R, then $H^i_I(M) \cong H^i_{IS}(M)$ as S-modules.