Commutative Algebra Seminar: Lecture of March 6, 2003

We next want to study local cohomology with support in a maximal ideal m of R. If M is any module such that every element is killed by a power of m, then for each $u \in M$, for some $N \in \mathbb{N}$ we have that Ru is a module over $R/m^N \cong R_m/m^N \cong \widehat{R_m}/m^N\widehat{R_m}$. (Since $\widehat{R_m}$ is the same as the m-adic completion of R, from now on we shall simply write $\widehat{R}$ for it.) This means that M is automatically a module over both R_m and $\widehat{R}$. We have $R_m \otimes_R M \cong M$ and $\widehat{R} \otimes_R M \cong M$ in this case. If S is flat over R, then $S \otimes_R \_$ is exact. If $I = (\underline{f})R$, viewing $H^\bullet_I(M)$ as $H^\bullet\big(\mathcal{K}(\underline{f}^\infty;\, M)\big)$ we see that

$$S \otimes_R H^\bullet_I(M) \cong S \otimes_R H^\bullet\big(\mathcal{K}^\bullet(\underline{f}^\infty;\, M)\big) \cong H^\bullet\big(S \otimes_R \mathcal{K}^\bullet(\underline{f}^\infty;\, M)\big) \cong$$

$$H^\bullet\big(\mathcal{K}^\bullet(\underline{f}^\infty;\, S \otimes_R M)\big) \cong H^\bullet_{IS}(S \otimes_R M).$$

We can apply this when $I = m$ and S is R_m or $\widehat{R_m} = \widehat{R}$ to conclude that $H^i_m(M)$ may be identified with $H^i_{mR_m}(M_m)$ and with $H^i_{m\widehat{R}}(\widehat{R} \otimes_R M)$ as well. When M is finitely generated, $\widehat{R} \otimes_R M \cong \widehat{M}$.

Thus, the study of local cohomology with support in a maximal ideal can be reduced to the local or even the complete local case. To go further, we need better understanding of injective hulls of residue fields.

Lemma. *If E is injective over any ring R then for every homomorphism $R \to S$ and every flat module G over S, $Hom_R(G,\, E)$ is injective over S.*

Proof. By the adjointness of tensor and Hom,

$$\mathrm{Hom}_S(M, \mathrm{Hom}_R(G,\, E)) \cong Hom_R(M \otimes_S G,\, E).$$

Since both $\_ \otimes_S G$ and $\mathrm{Hom}_R(\_,\, E)$ are exact functors, so is their composition. $\square$

Theorem. *Let $(R,\, m,\, K)$ be a local ring and let $E = \mathcal{E}_R(K)$ denote the injective hull of the residue class field. Let $^\vee$ denote $Hom_R(\_,\, E)$, which is evidently a contravariant exact functor.*

(a) *$K^\vee \cong K$, where $K = R/m$.*
(b) *If $M \neq 0$ then $M^\vee \neq 0$, i.e., $^\vee$ is faithfully exact.*
(c) *If M has finite length then $M^\vee$ has the same length.*
(d) *If J is any ideal $\subseteq m$, then $Ann_E J \cong Hom_R(R/J,\, E)$ is $\mathcal{E}_{R/J}(K)$.*
(e) *If R is Artinian then then E has finite length, and has the same length as R. Moreover the obvious map $R \to Hom_R(E,\, E)$ (sending r to multiplication by r) is an isomorphism.*
(f) *If R is a complete local ring then the obvious map $R \to Hom_R(E,\, E)$ is an isomorphism. More generally, if R is any local ring, then E is a module over $\widehat{R}$ in a unique way, and the obvious map $\widehat{R} \to Hom_R(E,\, E)$ is an isomorphism.*

(g) *If R is Artinian local then R is injective as an R-module if and only if R has a one-dimensional socle, in which case $R \cong E$.*

Proof. (a) We have an injection $K \hookrightarrow E$ that is essential. $K^\vee = \mathrm{Hom}_R(K, E) \cong \mathrm{Ann}_E m \subseteq E$ contains this copy of K, call it V_1, and is vector space V over K that is an essential extension of V_1. But since V is a vector space, an element not in V_1 cannot have a nonzero multiple in V_1. Thus, we must have $V = V_1$.

(b) Choose a nonzero element $u \in M$. Then $Ru \cong R/I$ where $I \subseteq m$, and we get a nonzero map $Ru \cong R/I \twoheadrightarrow R/m = K \subseteq E$. Since E is injective, this map extends to M.

(c) Use induction on the length $L = \ell(M)$ of M. We can choose a short exact sequence $0 \to K \to M \to M_1 \to 0$ where $\ell(M') = L - 1$. This yields $0 \to M_1^\vee \to M^\vee \to K^\vee \to 0$ where $K^\vee \cong K$ by (a). Then $\ell(M^\vee) = \ell(M_1^\vee) + 1 = (L-1)+1$ by the induction hypothesis.

(d) $\mathrm{Hom}_R(R/J, E)$ is injective over R/J by the Lemma above, and, since it embeds in E, it is an essential extension of K.

(e) Since $R^\vee \cong E$, the first statement follows from (c). The length of $\mathrm{Hom}_R(E, E)$ is the same as the length of E, which is the same as the length of R. Since the two modules have the same length, if the map $R \to \mathrm{Hom}_R(E, E)$ is injective it is surjective as well. But if $r \in R - \{0\}$ kills E, then $\mathrm{Hom}_R(R/rR, E) = E$ is the injective hull of K over R/rR as well as over R, while $\ell(R/rR) \neq \ell(R)$.

(f) Since every endomorphism of E stabilizes $E_t = \mathrm{Ann}_E m^t$ (which is $\mathcal{E}_{R/m^t}(K)$), and E is the increasing union of the E_t, the endomorphisms of E are given by the inverse limit over t of the modules $\mathrm{End}\,(E_t) \cong R/m^t$ by part (e). This inverse limit is $\widehat{R} \cong R$.

(g) If R is injective, it must be a direct sum of copies of E: every injective module is. Since E has the same length as R, we must have only one copy, i.e., $R \cong E$. Clearly, if $R \cong E$ the socle is one-dimensional, since this is true for E. If the socle V in R is one-dimensional, the injection $V \cong K \to E$ extends to R. Since the R is an essential extension of V, this gives an injection of R into E. Since they have the same length, we must have $R \cong E$. $\square$

Note that if M, W are any two R-modules, there is a canonical map

$$M \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(M, W), W\big)$$

that sends $u \in M$ to the map whose value on $f \in \mathrm{Hom}_R(M, W)$ is $f(m)$. In particular, if R is local, there is a canonical map $M \to M^{\vee\vee}$, and we make use of this tacitly below.

Corollary (duality theorem). *If (R, m, K) is a local local ring then $E = \mathcal{E}_R(K)$ has DCC, and a module has DCC if and only if, equivalently:*
(1) *it is embeddable in a finite direct sum of copies of E*
(2) *every element is killed by a power of m, and the socle is a finite-dimensional vector space over K.*

Moreover, if N has DCC then the natural map $N \to N^{\vee\vee}$ is an isomorphism, and if M has ACC, the natural map $M \to M^{\vee\vee}$ may be identified with $M \to \widehat{M}$. If R is complete, *this means that, with $^\vee$ used as the map in both directions, the categories of modules with ACC and modules with DCC are anti-equivalent.*

Proof. If E has an infinite strictly descending chain of submodules then applying $^\vee$ gives a sequence of surjections from $\widehat{R}$ whose kernels are strictly increasing, contradicting the fact that $\widehat{R}$ is Noetherian. Clearly, any submodule of a finite direct sum of copies of E has DCC. Now suppose that N has DCC. If P were an associated prime other than m, then $R/P \hookrightarrow N$, and R/P is Artinian: this shows we must have had $P = m$. Thus, every element of N is killed by a power of m. The socle is clearly a finite-dimensional vector space, since it is a vector space with DCC. Thus, it suffices to show that (2) implies (1). Let the socle be $K^{\oplus h}$. Taking the direct sum of h copies of $K \hookrightarrow E$, we get $\mathrm{Soc}\, N \hookrightarrow E^{\oplus h}$. The latter is injective and so the map extends to N. It is injective on N because N is an essential extension of its socle.

Now suppose N has DCC. By the argument in the preceding paragraph, we can give an injection $N \hookrightarrow E^{\oplus h}$. The cokernel C has DCC, and so injects into $E^{\oplus k}$. This yields a short exact sequence $0 \to N \to E^{\oplus h} \to E^{\oplus k}$. Since each $\mathrm{Hom}_R(E,\, E)$ may be identified with $\widehat{R}$, the map is given by a $k \times h$ matrix A with entries in $\widehat{R}$. Dualizing yields the sequence $\widehat{R}^{\oplus k} \to \widehat{R}^{\oplus h} \to N^\vee \to 0$ where the map is given by the transpose of A. Dualizing again transposes the matrix again, proving that $N \cong N^{\vee\vee}$. Similarly, if M has ACC we may choose a short exact sequence $R^{\oplus k} \to R^{\oplus h} \to M \to 0$ where the map is given by a matrix A' over R. Dualizing gives $0 \to M^\vee \to E^{\oplus h} \to E^{\oplus k}$ where the map is given by the transpose of A', and dualizing again yields $\widehat{R}^{\oplus k} \to \widehat{R}^{\oplus h} \to M^{\vee\vee} \to 0$ where the map is given by A'. This yields the isomorphism $M^{\vee\vee} \cong \widehat{R} \otimes_R M$. It is not difficult to see that, in both directions, we have an isomorphism of functors (in one case with the identity functor and in the other case with the completion functor). $\square$

There is a close connection between the study of the modules $H^i_m(M)$ and the study of injective hulls of residue fields. The first main indication of this is the following:

Theorem. *Let $(R,\, m,\, K)$ be a Gorenstein local ring of Krull dimension d. Then $H^d_m(R)$ is an injective hull of K. In particular, this holds when R is regular.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters. $H^d_m(R)$ is the direct limit of the modules $N_t = R/(x_1^t,\, \ldots,\, x_d^t)R$, where the map from N_t to N_{t+1} is induced by multiplication by $x_1 \cdots x_d$. Because $x_1, \ldots, x_d$ is a regular sequence in R, these maps are injective. The ring N_t has a one-dimensional socle, of which it is an essential extension. Since the maps $N_t \to N_{t+1}$ is injective, the socle in N_t must map bijectively to the socle in N_{t+1}. It follows that $H^d_m(R)$ is also an essential extension of K. Thus, $H^d_m(R)$ embeds in E. If the two are different, then for some t the annihilator of $I_t = (x_1^t,\, \ldots,\, x_d^t)R$ in E will be larger than its annihilator in $H^d_m(R)$. But the former may be identified with $\mathcal{E}_{R/I_t}(K)$, which has the same length as R/I_t (and, in fact, is isomorphic with it), and the latter contains $N_t = R/I_t$, which has already achieved the needed length. $\square$