Commutative Algebra Seminar: Lecture of April 3, 2003

Remarks. This discussion is intended for those with some familiarity with the theory of schemes. Let R be a Noetherian ring, let $X = \mathrm{Spec}\ R$, as a scheme, and let U be the open subset of X that is the complement of $V(I)$, where $I = (f_1, \ldots, f_n)R$ is an ideal of R and $V(I) = \{P \in \mathrm{Spec}\ R : I \subseteq P\}$. Let M be any R-module and $M^\sim$ the corresponding sheaf on $\mathrm{Spec}\, R$ whose module of sections on the open set $D_f \cong \mathrm{Spec}\ R_f$ is M_f. Note that U is also the union of the D_{f_i}. Then $H^0(X, M^\sim) \cong M$ while $H^i(X, M^\sim) = 0$ for $i > 0$, because X is an affine scheme. $H^i(U, M^\sim)$ is the cohomology of the Cech complex

$$0 \to \bigoplus_j M_{f_j} \to \bigoplus_{j_1, j_2} M_{f_{j_1} f_{j_2}} \to \cdots \to M_{f_1 \cdots f_n} \to 0$$

which is the same as the complex obtained by dropping the term numbered 0 from $\mathcal{K}^\bullet(f^\infty;\, M)$ and renumbering so that the degree i term in the Cech complex is the same as $\mathcal{K}^{i+1}(f^\infty;\, M)$ for $i \geq 0$. From this one can see that there is a long exact sequence:

$$0 \to H^0_I(M) \to H^0(X,\, M^\sim) \to H^0(U,\, M^\sim) \to H^1_I(M)$$
$$\to H^1(X,\, M^\sim) \to H^1(U,\, M^\sim) \to H^2_I(M) \to H^2(X,\, M^\sim) \to \cdots$$

from which one has that $H^i_I(M) \cong H^{i-1}(U,\, M^\sim)$ for $i \geq 2$. The first part of the sequence shows that the image of an element of $H^0(U,\, M^\sim)$ in $H^1_I(M)$ represents the obstruction to extending the section on U to a section defined everywhere on X. This is one of the reasons for the name "local cohomology" (think of the case where $V(I)$ is just one point, i.e. where I is primary to maximal ideal).

Theorem. *Let $(R,\, m,\, K)$ be a local ring of Krull dimension d.*
(a) *If R is Gorenstein, then for all i, $H^i_m(M) \cong \mathit{Tor}^{d-i}(M,\, H^d_m(R))$.*
(b) *For every finitely generated R-module M, $H^i_m(M)$ has DCC over R.*

Proof. (a) The complex $\mathcal{K}^\bullet(\underline{x}^\infty;\, R)$ is a flat resolution (numbered backwards) of $H^d_m(R)$: its cohomology for $i < d$ is zero because it is a direct limit of Koszul cohomology

$$H^i(x_1^t,\, \ldots,\, x_d^t;\, R) \cong H_{d-i}(x_1^t,\, \ldots,\, x_d^t;\, R),$$

which vanishes for $i < d$ because $x_1, \ldots, x_d$ is a regular sequence on R. The result follows because flat resolutions may be used to compute Tor, and $H^\bullet_m(M)$ is the cohomology of $H^\bullet_m(R) \otimes_R M$.

(b) The issues are unaffected if we replace R, M by $\widehat{R}$, $\widehat{M}$. Thus, we may assume that R is complete. But then R is a homomorphic image of a complete regular local ring $(S,\, n)$ where n maps onto m. Then $H^i_n(M) \cong H^i_m(M)$. Thus, there is no loss of generality in assuming that R is complete regular. Use a free resolution of M by finitely generated free R-modules to compute $\mathrm{Tor}^\bullet_R\big(M,\, H^d_m(R)\big)$. When one tensors the complex with $H^d_m(R) \cong E$ one gets a complex in which every term is a finite direct sum of copies of E. Thus, every term has DCC, and the homology has DCC. $\square$

Theorem. *Let $(R,\, m,\, K)$ denote a local ring of Krull dimension d. Let M be an R-module.*

(a) $H^d_m(M) = M \otimes_R H^d_m(R)$. *If R is Gorenstein, $H^d_m(M) = M \otimes_R \mathcal{E}_R(K)$.*

(b) *(local duality for Gorenstein local rings) If M is finitely generated, R is Gorenstein, $E = \mathcal{E}_R(K)$, and $^\vee$ denotes $Hom_R(\_\,,\, E)$, then $H^i_m(R) = Ext^{d-i}_R(M,\, R)^\vee$ for all $i \in \mathbb{Z}$.*

Proof. For part (a), if $x_1,\, \ldots\,, x_d$ is a system of parameters for R, we have that $H^d_m(M) \cong H^d_{(x_1,\, \ldots\,, x_d)R}(M) \cong H^d_{(x_1,\, \ldots\,, x_d)R}(R) \otimes_R M \cong E \otimes_R M$.

For (b), compute Ext using a free resolution of M, $G_\bullet$, which we may assume consists of finitely generated free modules. Then $\mathrm{Ext}^{d-i}_R(M,\, R)^\vee = H^{d-i}\big(\mathrm{Hom}_R(G_\bullet,\, R)\big)^\vee = H^{d-i}\big(\mathrm{Hom}_R\big(\mathrm{Hom}_R(G_\bullet,\, R),\, E\big)\big)$. For G finitely generated and free, we have an isomorphism of $G \otimes_R E \to \mathrm{Hom}_R\big(\mathrm{Hom}_R(G,\, R),\, E\big)$ coming from the bilinear map on $G \times E$ whose value on $(g,\, e)$ is the map that sends f to $f(g)e$. This is easily checked to be an isomorphism when $G = R$, and the construction commutes with direct sum. The isomorphism is functorial in the free module G. Thus $H^{d-i}\big(\mathrm{Hom}_R(\mathrm{Hom}_R(G_\bullet,\, R),\, E)\big) \cong H^{d-i}(G_\bullet \otimes_R E) = \mathrm{Tor}^R_{d-i}(M,\, E)$, and the result now follows from part (a) of the preceding theorem. $\square$

It is of interest to describe what the smallest and largest values of i are such that $H^i_m(M) \neq 0$. The result for the smallest works for any ideal, not just maximal ideals. Also note that this gives a different proof that $H^i_m(R) = 0$ for $i < \dim R$ for any Gorenstein (or Cohen-Macaulay) local ring $(R,\, m)$.

Proposition. *Let I be an ideal of a Noetherian ring R and let M be a finitely generated R-module. If $IM = M$, then $H^i_I(M) = 0$ for all i. If $IM \neq M$ and d is the depth of M on I, then $H^i_I(M) = 0$ for $i < d$ while $H^d_I(M) \neq 0$.*

Proof. Let $J = \mathrm{Ann}_R M$. Then $IM = M$ iff $I + J = R$. In this case, every element of every $H^i_I(M)$ is killed by both a power of I and J, and hence by R.

Now suppose that $IM \neq M$. Then the depth d of M on I is finite, and we may use induction on d. If $d = 0$, there is an element of M killed by I, and it follows that $H^0_I(M) \neq 0$. If $d > 0$ choose $x \in I$ not a zerodivisor on M and consider the long exact sequence for local cohomology coming from the short exact sequence $0 \to M \xrightarrow{x\cdot} M \to M/xM \to 0$:

$$\cdots \to H^{j-1}_I(M/xM) \to H^j_I(M) \xrightarrow{x\cdot} H^j_I(M) \to H^j_I(M/xM) \to \cdots$$

The depth of M/xM on I is $d-1$, and so by the induction hypothesis, $H^j_I(M/xM) = 0$ for $j < d-1$, which implies that multiplication by x is injective on $H^j_I(M)$ for $j < d$. But every element of every $H^i_I(M)$ is killed by a power of I and so by a power of x. It follows that $H^i_I(M) = 0$ for $j < d$. But then the long exact sequence implies that $H^{d-1}_I(M/xM)$ injects into $H^d_I(M)$, and so the latter is not zero. $\square$

Proposition. *Let $(R,\, m,\, K)$ be a local ring and let M be a finitely generated R-module. Let d be the Krull dimension of M. Then $H^d_m(M) \neq 0$, while $H^i_m(M) = 0$ for $i > d$.*

Proof. We may replace R by R/J where $J = \mathrm{Ann}_R M$ without affecting the statement. Thus, we may assume without loss of generality that M is faithful and that $\dim R = d =$

$\dim M$. In computing local cohomology we may replace m by the ideal $(x_1, \ldots, x_d)R$ generated by a system of parameters for R. It follows that $H^i_m(M) = 0$ for $i > d$.

It remains to prove that $H^d_m(M) \neq 0$. The statement is unaffected if we replace R and M by their completions. Thus, we may assume that R is complete. Second, we may choose N maximal in M such that M/N has dimension d. Then $H^d_m(M) \to H^d_m(M/N)$ is onto, so we may assume that every proper quotient of M has dimension smaller than d. Again, we may assume as well that M is faithful over R. Then R embeds in a finite direct sum of copies of M, and so $\mathrm{Ass}\,(R) \subseteq \mathrm{Ass}\,(M)$. If $\mathrm{Ass}\,(R)$ has two or more primes, so does M. We show that M cannot have two associated primes. Choose one, P, to be such that $\dim R/P = d$, and any other, Q. Then $N = R/Q$ embeds in M, P must be an associated prime of M/N, so that $\dim M/N = d$, a contradiction. It follows that both $\mathrm{Ass}\,(R)$ and $\mathrm{Ass}\,(M)$ have a unique associated prime P, and so P must consist of all nilpotent elements of R. Then M/PM has the same dimension as M, and so $PM = 0$. Since M is faithful, $P = 0$. Thus, we may assume that R is a complete domain and that M is torsion-free. Then R is module-finite over a regular local ring A, and the maximal ideal of A expands to an ideal whose radical is m. It follows that we may replace R by A. We have therefore reduced to the case where R is regular and M is torsion-free. Then M can be embedded in a nonzero finitely generated free R-module G so that the quotient is torsion, and so has dimension smaller than R. Then $H^d_m(M)$ maps onto $H^d_m(G)$, which is a nonzero direct sum of copies of $H^d_m(R)$, and is not zero. □

A module M over a domain R is called *solid* if there is a nonzero R-linear map f from M to R, i.e., if $\mathrm{Hom}_R(M, R) \neq 0$. Note that the image of f will be a nonzero ideal of R. If S is an R-algebra that is solid when considered as an R-module, we say that S is a *solid* R-algebra. Note that there is no finiteness condition on M here. If S is a solid R-algebra, there is an R-linear map $S \to R$ that takes a nonzero value on $1 \in S$, for if $g : S \to R$ and $g(s_0) \neq 0$, we may use instead the map whose value on s is $g(ss_0)$.

Theorem. *Let R be a Noetherian domain of characteristic $p > 0$ and let $N \subseteq M$ be finitely generated R-modules. Let $u \in M$. If there is a solid R-algebra S such that $1 \otimes u \in S \otimes_R M$ is in the image of $S \otimes_R N$ in $S \otimes_R M$, then $u \in N^*_M$.*

Proof. It suffices to consider the case where M is replaced by a finitely generated free module G that maps onto it, N by its inverse image G, and u by an element of G that maps to u. Thus, we may assume that M is free, say $M = R^h$, and we may identify $S \otimes_R M$ with S^h. We may think of the image if $S \otimes_R N$ in $S \otimes_R N$ as the "expansion" NS of N to S^h. We have that $u \in NS$. We let Frobenius act coordinatewise on S^h. Then $u^q \in (NS)^{[q]} = (N^{[q]})S$. Let $g : S \to R$ be R-linear such that $g(1) = c \neq 0$. Applying g coordinatewise, we find that $cu^q \in N^{[q]}$ for all q. □

We want to prove that the converse holds when R is a complete local domain. We shall need to expend a considerable effort to do this. The proof depends on knowing the result on the Cohen-Macaulay property for R^+.