Commutative Algebra Seminar: Lecture of April 10, 2003

We next note the following fact:

Theorem. *Let (R, m, k) be a complete local domain. Then M is solid if and only if $H^d_m(M) \neq 0$.*

Proof. If M is solid we have $M \twoheadrightarrow I \subseteq R$, and a surjection $H^d_m(M) \twoheadrightarrow H^d_m(I)$. As a module, $\dim I = d$, and so $H^d_m(I) \neq 0$. It follows that $H^d_m(M) \neq 0$.

Now suppose that $H^d_m(M) \neq 0$. Note that R is module-finite over a complete regular local ring A with the same residue class field, and that the maximal ideal m_0 of A expands to an ideal primary to m in R. Thus,

$$H^d_m(M) = H^d_{m_0}(M) \cong H^d_{m_0}(A) \otimes_A M = \mathcal{E}_A(K) \otimes_A M \neq 0.$$

We reverse the order of the factors in this tensor product, and note as well that, the dual of $M \otimes_A \mathcal{E}_A(K)$ into $\mathcal{E}_A(K)$ over A is nonzero, i.e.,

$$0 \neq \mathrm{Hom}_A(M \otimes_A \mathcal{E}_A(K),\, \mathcal{E}_A(K)) \cong \mathrm{Hom}_A\big(M,\, \mathrm{Hom}_A\big(\mathcal{E}_A(K),\, \mathcal{E}_A(K)\big)$$

by the adjointness of tensor and Hom, and

$$\mathrm{Hom}_A\big(\mathcal{E}_A(K),\, \mathcal{E}_A(K)\big) \cong A$$

by our duality theory, since A is complete. Thus, $\mathrm{Hom}_A(M,\, A) \neq 0$, and we can choose a nonzero map $g : M \to A$. Suppose that $u \in M$ is such that $g(u) = a \in A - \{0\}$. We apply $\mathrm{Hom}_R(R,\, \_\,)$ to get a map $\mathrm{Hom}_A(R,\, M) \to \mathrm{Hom}_A(R,\, A)$: this map is easily checked to be R-linear (the R-module structure on $\mathrm{Hom}_A(R,\, M)$ is induced by the R-module structure on the copy of R), and there is an R-linear injection $M \to \mathrm{Hom}_A(R,\, M)$ that sends each $v \in M$ to the map taking r to rv for all $r \in R$. The composite map

$$M \to \mathrm{Hom}_A(R,\, M) \to \mathrm{Hom}_A(R,\, A)$$

is therefore R-linear, and sends u to a map whose value on $1 \in R$ is $g(u) = a \neq 0$. This gives a nonzero R-linear map of M to $\mathrm{Hom}_A(R,\, A)$, and the argument is complete once we note that this is a torsion-free R-module of rank one, and so embeds in R. It is clearly torsion-free, since there is no A-torsion and every nonzero element of R has a nonzero multiple in A. The rank can be computed after localizing at the nonzero elements of A: A becomes a field, R a finite field extension, and the result follows because then R and $\mathrm{Hom}_A(R,\, A)$ have the same vector space dimension over A. $\square$

Proposition. *Let (R, m, K) be a complete local domain. Then any big Cohen-Macaulay algebra B for R is solid.*

Proof. Let $x_1, \ldots, x_d$ be a system of parameters for R. Then $x_1, \ldots, x_d$ is a regular sequence on B. We may view $H^d_m(B)$ as the direct limit of the modules $B_t = B/(x_1^t, \ldots, x_d^t)B$, where the map from $B_t \to B_{t+1}$ is induced by multiplication by the element $x_1 \cdots x_d$. The R-sequence property implies that the maps in the direct limit system are injective, i.e., that $(x_1^{t+1}, \ldots, x_d^{t+1})B :_B x_1 \cdots x_d = (x_1^t, \ldots, x_d^t)B$. We leave this an exercise. But then $0 \neq B/(x_1, \ldots, x_d)B$ injects into $H^d_m(B)$. $\square$

The following result is essentially in [M. Hochster, *Solid closure*, in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159**, Amer. Math. Soc., Providence, R. I., 1994, 103–172], although it is not explicitly stated there in quite as strong a form as that given here.

Theorem. *Let R be a complete local domain of prime characteristic $p > 0$. Then there is a quasilocal R^+-algebra B such that (1) $R^+ \to B$ is local and (2) for every module-finite extension R_1 of R with $R \subseteq R_1 \subseteq R^+$, B is a big Cohen-Macaulay algebra for R_1 with the property that for every pair of finitely generated R_1-modules $N \subseteq M$ and $u \in M$, $u \in N^*_M$ if and only if $1 \otimes u$ in $B \otimes_R M$ is in the image of $B \otimes_R N$ in $B \otimes_R M$. In particular, for ideals $I \subseteq R_1$, $u \in I^*$ if and only if $u \in IB \cap R_1$.*

Proof. First suppose that B has all the properties specified except that it need not be quasilocal. Then we may localize at prime containing mB to get a choice of B with $R \to B$ local. B is still a big Cohen-Macaulay algebra, and therefore solid. Therefore the specified condition for being in the tight closure is still sufficient.

B is constructed as a very large direct limit of algebras obtained from R^+ by two sorts of algebra modification. One eliminates relations on parameters for one choice of R_1 in the usual way, and the other forces elements that are in tight closures into expansions. It suffices to show that if one does finitely many such modifications to obtain an extension ring S_r of R^+, one does not have $1 \in mS_r$. All of the systems of parameters, instances of tight closure, etc. involve only finitely many elements from R^+, and one sees that if there is a counterexample of this sort, it arises from a single choice of R_1. Thus, it suffices to show that for a single choice of R_1, after finitely many modifications of the specified sort, one does not have $1 \in mS_r$. We may change notations so that $R = R_1$.

By taking a direct sum of modules, the finitely many instances of tight closure with which we are concerned can be combined into a single instance. By mapping a finitely generated free module G onto the module, we may assume that we have $u \in G$ (where G is free) such that $u \in H^*$, where $H \subseteq G$: say that H is spanned by finitely many vectors $w_1, \ldots, w_s$. Let $z_1, \ldots, z_s$ be finitely many indeterminates. Form the ring $R[z_1, \ldots, z_s]/I$, where I is generated by the entries of $u - z_1 w_1 - \cdots - z_s w_s$.

Call the ring obtained this way S_0. In addition, perform several algebra modifications for relations on parameters. This means, at the i th stage, that one has a relation $x_{k+1}u = \sum_{j=1}^k x_j u_j$ where $u, u_1, \ldots, u_k \in S_i$, and $S_{i+1} = S_i[y_1, \ldots, y_k]/(u - \sum_{j=1}^k y_j x_j)$. The problem is that after some sequence of modifications like this, one may have that the image of 1 in S_r is in mS_r.

If this happens, it also happens with *partial algebra modifications* of degree N for some large N. In a partial algebra modification one only adjoins the monomials of total degree at most N in the variables, and instead of killing the entire ideal specified, one only kills linear combinations of the specified generator(s) whose coefficients have total degree at most $N-1$ in the variables. By a direct limit argument, if $1 \in mS_r$ for some sequence of algebra modifications of the kinds specified, this also happens for partial algebra modifications

$$R \to T_0 \to T_1 \to \cdots \to T_r$$

of some large degree $N \gg 0$. While r and N may be large, they are now fixed in the rest of the argument. We shall obtain a contradiction.

Since $u \in N^*$ we have a fixed $c \in R - \{0\}$ such that $c^{1/q}u \in \sum_{j=1}^{s} R^{1/q}w_j$, and so we certainly have that $u \in \sum_{j=1}^{s}(R^+ \cdot c^{-1/q})w_j$. Using the values in $R^+ \cdot c^{-1/q}$ for the indeterminates z_j, we get a commutative diagram:

$$\begin{array}{ccc} R & \longrightarrow & R^+ \cdot c^{-N/q} \\ \uparrow & & \uparrow \\ R & \longrightarrow & T_0 \end{array}$$

where the leftmost map is the identity and the other vertical map is R-linear.

At the next stage one uses that R^+ is big Cohen-Macaulay algebra to show that there are values for the y_i used in forming T_1 in $R^+ \cdot c^{-N/q}$ to which one may map the y_i and kill the relation. The monomials in the y_i that are used have degree at most N, and so they map into $R^+ \cdot c^{-N^2/q}$, but one has coefficients as well that are in $R^+ \cdot c^{-N/q}$. It follows that there is a commutative diagram:

$$\begin{array}{ccccc} R & \longrightarrow & R^+ \cdot c^{-N/q} & \longrightarrow & R^+ \cdot c^{-(N+N^2)/q} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & T_0 & \longrightarrow & T_1 \end{array}$$

where the leftmost map is the identity and the other vertical maps are R-linear.

One can evidently continue in this way: T_i maps to $R^+ \cdot c^{-N_i}/q$, where $N_0 = N$ and $N_{i+1} = NN_i + N_i = (N+1)N_i$. The key point is that N_r does not depend on q. Looking at the extremes only, for all $q \gg 0$ we have a commutative diagram

$$\begin{array}{ccc} R & \longrightarrow & R^+ \cdot c^{-N_r/q} \\ \uparrow & & \uparrow \\ R & \longrightarrow & T_r \end{array}$$

where the leftmost map is the identity and the other vertical map is R-linear. Since $1 \in mT_r$, supposedly, we find that for all $q \gg 0$, $1 \in mR^+ \cdot c^{-N_r/q}$, i.e., that $(c_r^N)^{1/q} \in mR^+$ for all $q \gg 0$. One can choose a discrete valuation ord on R that is positive on m. This extends to R^+, with values in the rational numbers allowed. The value on any generator of m is at least 1, and so the value of any element in mR^+ is at least 1. But $\mathrm{ord}\left((c^{N_r})^{1/q}\right) = (1/q)\mathrm{ord}\,(c^{N_r}) \to 0$ as $q \to \infty$. This is a contradiction. $\square$