Commutative Algebra Seminar: Lecture of April 17, 2003

We now know that over a complete local domain of characteristic $p > 0$, an element is in the tight closure of a submodule of a free module iff it is in the contracted expansion from some solid algebra. If $I \subseteq R$ with $I = (f_1, \ldots, f_s)R$, $u \in R$ and $Z_1, \ldots, Z_s$ are indeterminates over R, we shall say that $S = R[Z_1, \ldots, Z_s]/(u - Z_1 f_1 - \cdots - Z_s f_s)$ is a *generic forcing algebra* over R for the ideal $I \subseteq R$ and the element $u \in R$. More generally, if $H \subseteq G = R^h$ and $u = (u_1, \ldots, u_h) \in R^h$, while H is spanned by $f_1, \ldots, f_s$, we shall say that $S = R[Z_1, \ldots, Z_s]/J$ where J is spanned by the entries of the vector $u - Z_1 f_1 - \cdots - Z_s f_s$ is a *generic forcing algebra* over R for the inclusion $H \subseteq G$ and the element $u \in G$.

Note that if there exists any R-algebra T such that $u \in IT$ (respectively, $u \in HT$) and S is a generic forcing algebra as above, then the map $R \to T$ factors $R \to S \to T$. One simply sends the Z_j to the coefficients $y_j \in T$ such that $u = y_1 f_1 + \cdots + y_s f_s$. From this it follows that one can test whether an element is in a tight closure with a generic forcing algebra:

Theorem. *Let R be a complete local domain of characteristic $p > 0$ of Krull dimension d. Let $I \subseteq R$ and let $u \in R$ (or let G be a finitely generated free R-module, $H \subseteq G$, and $u \in G$). Let I (or H) have generators $f_1, \ldots, f_s$. Let S be the generic forcing algebra as described just above. Then $u \in I^*$ (respectively, $u \in H^*_G$) if and only if $H^d_m(S) \neq 0$, i.e., if and only if S is a solid R-algebra.*

Proof. The generic forcing algebra S has the property that u is in the required expansion. Hence, if S is solid, then u is in the tight closure. On the other hand, if u is in the tight closure, then there is some solid algebra T such that u is in the expansion of I (or H) to T. Then S maps to T. Since T is solid there is some R-linear map $T \to R$ with nonzero value on $1 \in T$. But then S is solid, since the composite $S \to T \to R$ has nonzero value on $1 \in S$. $\square$

There are similar results in which the solid algebra is replaced by a module on which Frobenius acts. The notion of being solid needs modification.

Let R be a ring of prime characteristic $p > 0$ and let M be an R-module. We shall say the M has a Frobenius action F if we have a $\mathbb{Z}$-linear map $F : M \to M$ such that for all $r \in R$ and $m \in M$, $F(rm) = r^p F(m)$. Another way of thinking about modules with Frobenius action is this: we may construct a *non-commutative* ring $R\langle F\rangle$ by adjoinging the non-commutative indeterminate F to R and imposing the relations that $Fr = r^p F$ for all $r \in R$. Every element of $R\langle F\rangle$ can be written as a polynomial in F of the form $\sum_{j=0}^{n} r_j F^j$. Thus, $R\langle F\rangle$ is free on the symbols F^j as a left R-module. To give a module with Frobenius action is the same thing as to give an $R\langle F\rangle$-module.

R itself with the usual action of the Frobenius endomorphism is an example. So is any R-algebra. The local cohomology modules $H^i_I(S)$ of any R-algebra S with support in any ideal are also examples. If one uses the complex $\mathcal{K}(\underline{f}^{\infty};\, S)$ to compute them, where the $\underline{f} = f_1, \ldots, f_n$ generate I up to radicals, this follows because each S_g is an R-algebra and so has a Frobenius action: the terms in the complex are direct sums of such algebras, and the maps induced by maps of algebras on the summands, possibly with signs attached. (One can verify independence of the Frobenius action from the choice of f_j.) We have:

Theorem. *Let R be a complete local domain of characteristic $p > 0$. Let $I \subseteq R$ and $u \in R$. then $u \in I^*$ if and only if there exists an R-module M with Frobenius action, an element $v \in M$ such that $F(v) = v$, and an R-linear map $g : M \to R$ such that $g(v) \neq 0$ and such that $uv \in IM$.*

Proof. If $u \in I^*$ we may choose M to be a solid R-algebra S such that $u \in IS$, taking $v = 1$.

For the converse, given such an M, apply F^e to $uv = \sum_{j=1}^h i_j m_j$ to get $u^{p^e} v = \sum_{j=1}^h i_j^{p^e} F^e(m_j)$, and then apply the map g to get that $g(v)u^{p^e} = I^{[p^e]}$ for all e. $\square$

There is an analogy to the situation for algebras in that given $I = (f_1, \ldots, f_s)R$ and $u \in R$, we can make a "generic" choice of M. Let $M = R \oplus R\langle F\rangle^{\oplus s}/R\langle F\rangle(u, -f_1, \ldots, -f_s)$ where the image of $1 \in R$ in the initial summand R of the numerator will play the role of v. This maps to any other choice of $R\langle F\rangle$-module M' with an element v such that $F(v) = v$ and such $uv \in IM'$. Note that if v is such that $F(v) = v$ then $F(rv) = r^p F(v) = r^p v$ for all $r \in R$, so that R with the usual Frobenius structure maps $R\langle F\rangle$-linearly to M' in such a way that 1 maps to v.

There is also a local cohomology criterion for there to exist an R-linear map $g : M \to R$ such that $g(v) \neq 0$. It is that the map $H^d_m(R) \to H^d_m(M)$ not be zero, with $d = \dim R$ (where the map $R \to M$ is induced by the fact that R is the initial direct summand of the numerator in the definition of M). If the composite $R \to M \to R$ is nonzero, it will be multiplication by some nonzero element $r \in R$, and the induced map $H^d_m(R) \to H^d_m(R)$ will be given by multiplication by r and not zero. (If R is module-finite over the complete regular local ring A, by local duality over A the dual of the top local cohomology module is $\mathrm{Hom}_A(R,\, A)$ which is torsion-free over A and hence over R.) Conversely, if the the map $H^d_m(R) \to H^d_m(M)$ is nonzero, view it as $R \otimes_A E \to M \otimes_A E$, where $E = \mathcal{E}_A(K) \cong H^d_{m_A}(A)$, dualize into E over A, and use the adjointness of $\otimes$ and Hom to obtain a nonzero map $\mathrm{Hom}_A(M,\, A) \to \mathrm{Hom}_A(R,\, A)$: this can also be obtained more directly from $R \to M$ by applying $\mathrm{Hom}_A(\_\,,\, A)$, and is nonzero. This means that the there is an A-linear map $f : M \to A$ such that $f(rv) \neq 0$ for some $r \in R$. Replacing f by rf, if necessary, we may assume that $f(v) \neq 0$. Now apply $\mathrm{Hom}_A(R,\, \_\,)$ to get an R-linear map $\mathrm{Hom}_A(R,\, M) \to \mathrm{Hom}_A(R,\, A)$ which is R-linear with the R-module structures coming from the copy of R in the first variable in the Hom. The composite $M \to \mathrm{Hom}_A(R,\, M) \to \mathrm{Hom}_A(R,\, A)$ (where the first map takes w to the map that sends r to rw) is then R-linear and nonzero on v. Finally, $\mathrm{Hom}_A(R,\, A)$ embeds in R.

Our next major objective is the following difficult result:

Theorem (K. E. Smith). *Let R be a locally excellent Noetherian domain of prime characteristic $p > 0$. If I is a parameter ideal in the sense that for every prime ideal P of R containing I, IR_P is generated by part of a system of parameters, then $I^* = I^+$.*

The proof is comprised of a sequence of steps: here is an overview.

(1) Reduce to the normal local case, where the local ring has lowest possible dimension. Simultaneously, reduce to the case where the system of parameters is a full system of parameters in the local ring.

(2) Reduce to the complete case. This needs a generalized form of the Artin approximation theorem, including some understanding of the nature of Henselization.

(3) Reduce to the Gorenstein case.

(4) Define 0^* and 0^+ in $H^d_m(R)$. The idea is that when one has a system of parameters $x_1, \dots, x_d$, one has that $R/(x_1, \dots, x_d)$ embeds in $H^d_m(R)$: in fact, when R is Gorenstein, the local cohomology may be viewed as an increasing union of modules $R/(x_1^t, \dots, x_d^t)R$. Thus, for an ideal I generated by a full system of parameters, I^+/I and I^*/I embed in $H^d_m(R)$. Taking a union over all possible choices of I, one gets the modules $0^+ \subseteq 0^*$ in $H^d_m(R)$.

(5) Use the minimality of the counter-example to show that $0^*/0^+$ in $H^d_m(R)$ has finite length. This involves showing that the test ideal for R is the annihilator of 0^* in $H^d_m(R)$ in the Gorenstein case, and that the formation of the test ideal commutes with localization in the Gorenstein case.

(6) From the condition in (5), develop a special sort of equation that an element $u \in R$ representing an element of $0^*/0^+$ must satisfy.

(7) Show that the special equation enables one to prove that $u \in I^+$.

We next want to carry through these steps. Before embarking on step (1), we prove two lemmas.

Lemma. *Let R be a Noetherian domain of characteristic $p > 0$ and let S be a module-finite extension domain. Let I be an ideal of R and $u \in R$. The $u \in I^*$ in R if and only if $u \in (IS)^*$ working over S. That is, passing to a module-finite extension domain does not affect whether an element is in a tight closure.*

Proof. If $u \in I^*$, it is obvious that this is preserved when we pass to any domain larger than R. Now suppose that $u \in (IS)^*$, so that $du^{p^e} \in (IS)^{[p^e]} = I^{[p^e]}S$ for all $e \gg 0$, with $d \in S - \{0\}$. First, note that a nonzero element of S has a nonzero multiple in R. Thus, we may replace d by a nonzero multiple c in R. Second, note that since S is torsion-free over R it embeds in a free R-module. One of the projection maps to R must be nonzero on the image of 1, so that we have $g : S \to R$ with $g(1) \neq 0$ (i.e., S is solid over R). Applying g we find that $g(1)u^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. $\square$

Lemma. *Let R be a locally excellent Noetherian domain of characteristic $p > 0$, and let I be a parameter ideal in the sense that I for all $P \supseteq I$, IR_P is generated by part of a system of parameters for R_P. Then every associated prime P of I^+ is a minimal prime of I and I^+, and IR_P is generated by a full system of paramters for R_P.*

Proof. Suppose that P is an associated prime of I. Then $I \subseteq P$. Since the formation of R^+ commutes with localization, we may assume that (R, P) is local. Then R is excellent, and I is generated by part of a system of parameters, $x_1, \dots, x_k$. Suppose that P contains an element y not in any minimal prime of I. We shall obtain a contridiction, which shows that P is a minimal prime of I and hence of I^+. Since P is an associated prime of I^+, and that remains true once we localize, P consists entirely of zerodivisors on R/I^+. It follows that for some $w \in R - I^+$, $yw \in I^+$. But $x_1, \dots, x_k, y$ is part of a system of parameters in R, and is therefore a regular sequence in R^+. Since $yu \in I^+$, we have that $yu \in IR^+$. Since $x_1, \dots, x_k, y$ is a regular sequence in R^+, we have that $u \in IR^+ \cap R = I^+$, a contradiction. $\square$

We are now ready to start carrying through the steps in the overview given earlier.

Step (1). If $u \in I^* - I^+$, then u fails to be in some primary component of I^+ (these are unique since the associated primes of I^+ are minimal by the lemma just above). Call the corresponding prime P. Since the elements of $R - P$ are not zerodivisors on that primary component, this is preserved when we localize at P. We have therefore reduced to the case where (R, m) is an excellent local domiain, and since P is minimal over I by the Lemma, we have that I is generated by a full system of parameters. Since R is excellent, its integral closure is module-finite over R. Thus, by the prior Lemma we also get a counter-example when we replace R by its normalization. Our counter-example may no longer by local, but we can localize again to restore that condition. Among all local, normal counter-examples (R, m) such that I is generated by a full system of parameters, we may choose one such that $d = \dim R$ is as small as possible. In particular, we may assume that for every proper localization of R, tight closure agrees with plus closure for parameter ideals. $\square$