Commutative Algebra Seminar: Lectures of April 24–25, 2003

Step 3. To reduce to the Gorenstein case, view the ring R as module-finite over $A = K[[x_1, \dots, x_d]]$, where $x_1, \dots, x_d$ is the system of parameters generating I. As before, let $u \in I^* - I^+$. Then replace R by $R_0 = A[u]$, which is evidently complete local: R is module-finite over R_0, and I is replaced by $(x_1, \dots, x_d)R_0$. Then $R^+ = R_0^+$, and, by an earlier lemma, we have $u \in (IR_0)^*$ over R_0. Thus, R_0 is also a counter-example, and, again, the Krull dimension has not changed. We claim that R_0 is Gorenstein. Consider the minimal monic polynomial $f(X)$ of u over the fraction field of A. All the roots of $f(X)$ are conjugate to u, and therefore satisfy any monic polynomial over A satisfied by u. But then the elementary symmetric functions of the roots are all integral over A as well as in the fraction field of A, and so $f(X)$ has all coefficients in A. Now, given any polynomial satisfied by u over A, one may apply the division algorithm *over* A to divide by the monic polynomial $f(X)$. The remainder, if nonzero, gives a polynomial of smaller degree over the fraction field of A satisfied by u, a contradiction. Therefore, the remainder must be 0. This shows that $R_0 = A[u] = A[X]/\big(f(X)\big)$. Thus, R_0 is a hypersurface and, therefore, Gorenstein. $\square$

Henceforth, we assume that our counter-example is an excellent local domain in which a full system of parameters gives a counter-example, that it has minimal Krull dimension among all such counter-examples, and that it is both complete and Gorenstein. However, for some purposes we shall also need to consider localizations of this domain at primes: completeness is then lost, but we still have an excellent Gorenstein domain, and if the prime is properly contained in the maximal ideal, the theorem may be assumed to be true for any such localization.

Note that for any local ring (R, m, K), if $E = \mathcal{E}_R(K)$, there is a bijection between submodules $N \subseteq E$ and surjections $\widehat{R} \twoheadrightarrow N^\vee$ obtained by applying the functor $^\vee = \mathrm{Hom}_R(\_\,, E)$. The surjection is completely determined by which ideal is killed. Since each of $N^\vee = \widehat{R}/J$ and N is the dual of the other, they have the same annihilator. Thus, every submodule of E is uniquely determined by its annihilator J, and the submodule may be identified with $(\widehat{R}/J)^\vee = \mathrm{Ann}_E J$. The submodules of E are thus in bijective correspondence with the ideals of $\widehat{R}$. Given an ideal of I of R, its annihilator in E is $\cong \mathcal{E}_K(R/I)$, and is the same as the annihilator of $\widehat{I} = I\widehat{R}$.

Now suppose that R is Gorenstein. Let $x_1, \dots, x_d$ be a system of parameters, and $I = (x_1, \dots, x_d)R$. The annihilator of I in E is an injective hull for R/I, and since R/I is Gorenstein, must be isomorphic to R/I itself. Thus, for every such I, $E \cong H^d_m(R)$ contains a unique copy of R/I (although it does not have a canonical choice of generator). If $I \subseteq J$ are two such parameter ideals, the copy of R/J in E is contained in the copy of R/I. We have already seen that if $J = I_t = (x_1^t, \dots, x_d^t)$ then the map from the copy of R/I to the copy of R/I_t, which is injective, is given by multiplication by $(x_1 \cdots x_d)^{t-1}$. The union of these is $H^d_m(R) \cong E$.

Step 4. We next want to show that if R is a local Gorenstein domain, then for every parameter ideal I, $I^+/I \subseteq I^*/I$ (this much we already know), and the latter *embeds* into $E \cong H^d_m(R)$. The union of these inclusions gives an inclusion of a pair of submodules of $H^d_m(R)$: we denote these by 0^+ and 0^*, respectively.

In an R-module B that need not be finitely generated, we can consider two notions of 0^*_B. One is the *finitistic* tight closure $0^{*\mathrm{fg}}{}_B$ of 0, and is simply the union of all the 0^*_N as N runs through the finitely generated submodules of B. It suffices to take the union over any directed family of finitely generated submodules of B whose union is B.

The other notion is obtained by applying the usual definition of tight closure directly to B itself: $b \in 0^*_B$ if there exists $c \in R^\circ$ such that for all $q = p^e \gg 1$, $cb^q = 0$ in $F^e(B)$.

In general, $0^{*\mathrm{fg}} \subseteq 0^*$. It is an important open question whether the two agree for Artinian modules over a reduced excellent local ring. An affirmative answer would imply that in the excellent case, weakly F-regular rings are F-regular. Special cases have been proved by G. Lyubeznik and K. E. Smith (e.g., the case of a graded Artinian module, and the case of an F-finite local domain with an isolated singularity) and by H. Elitzur.

We want to point out that these two notions agree in the highest local cohomology module of any excellent reduced equidimensional local ring.

Proposition (K. E. Smith). *IF R is a reduced equi-dimensional excellent local ring of Krull dimension d and of prime characteristic $p > 0$, then in $H^d_m(R)$, $O^{*\mathrm{fg}} = 0^*$.*

Proof. Fix a system of parameters $x_1, \ldots, x_d$, let $x = x_1 \cdots x_d$, and identify $H^d_m(R)$ with directed union of the R/I_t, as usual, where $I_t = (x_1^t, \ldots, x_d^t)R$, and suppose that the image of u in R/I_t represents an element of the tight closure of 0 in $H^d_m(R)$. Since we may replace $x_1, \ldots, x_d$ by $x_1^t, \ldots, x_d^t$, we might as well assume that we are working with the image of u in $R/(x_1, \ldots, x_d)R$. For some $c \in R^\circ$ and for all $q \gg 0$, we have that cu^q is 0 in $F^e\big(H^d_m(R)\big)$ which is the direct limit of the modules $F^e(R/I_t) = R/I_{qt}$ and may be identify with $H^d_m(R)$ again. This does not quite mean that cu^q is in $I^{[q]}$: it does mean that for this c and for all $q \gg 0$, there exists k_q such that $cx^{k_q}u^q \in I_{q+k}$, and so $cu^q \in (x_1^{q+k_q}, \ldots, x_d^{q+k_q}) : x^{k_q}$. By our earlier results on colon capturing for parameter ideals, we get that $cu^q \subseteq \big((x_1, \ldots, x_d)^{[q]}\big)^*$ for all $q \gg 0$, and multiplying by a test element c' yields that $c'cu^q \in I^{[q]}$ for all $q \gg 0$, as required. $\square$

From our earlier description of 0^* one sees that it is the finitistic tight closure of 0 in $H^d_m(R)$, but that means that it is also, simply, the tight closure of 0. If we fix one system of parameters $x_1, \ldots, x_d$ and consider the corresponding ideals I_t, then since $H^d_m(R)$ is the union of the R/I_t, we have that 0^* is the union of the I_t^*/I_t. The map from $I_t^*/I_t \to I^*_{t+k}/I_{t+k}$ is given by multiplication by x^k, and is injective.

It is easy to see that 0^+ is simply the kernel of the map $H^d_m(R) \to H^d_m(R) \otimes_R R^+$, and the latter may be identified with $H^d_m(R^+)$ and so with the direct limit of the modules R^+/I_tR^+. In the latter direct limit, the maps are injective because R^+ is a big Cohen-Macaulay algebra: the x_k form a regular sequence on it. We see that the maps $I_t^+/I_t \to I^+_{t+k}/I_{t+k}$ are likewise injective.

It follows that the result we are aiming to prove, which is that $I^+ = I^*$ for every parameter ideal, is equivalent to the vanishing of $0^*/0^+$: every I^*/I^+ embeds in it, and it is the union of these. Note that 0^+ can also be regarded simply as the union of I_t^+/I_t for a single system of parameters $x_1, \ldots, x_d$.

Step 5 of our program consists of showing that for a minimum Krull dimension complete local Gorenstein domain that is a counter-example, one has that $0^*/0^+$ has finite length. We postpone this step and save it for last. The next two steps show how to finish the proof of the result once we know this.

Step 6. Now suppose that we have shown that $0^*/0^+ \subseteq H_m^d(R)$ has finite length. Let u be an element of $I^* - I^+$. For every e, $u^{p^e} \in (I^{[p^e]})^* - (I^{[p^e]})^+$. Thus, the Frobenius images of $\overline{u} \in H_m^d(R)$ are all nonzero. Let N_e denote the R-span in $H_m^d(R)$ of $F^e(\overline{u})$, $F^{e+1}(\overline{u})$, $\ldots$. The N_e form a decreasing sequence of nonzero submodules of $H_m^d(R)$, which is eventually stable. After replacing u by u^{p^e} for sufficiently large e, we may assume that all of the N_e are equal. Think of $H_m^d(R)$ as the direct limit of the $R/(x_1^t, \ldots, x_d^t) = R/I_t$ where, with $x = x_1 \cdots x_d$, the map from $R/I_t \to R/I_{t+k}$ is induced by multiplication by x^k. Write $[v, I_t]$ for the class of $v \in R$ modulo I_t thought of as an element of $H_m^d(R)$, so that $[v, I_t] = [x^k v, I_{t+k}]$. Note that with this convention, $F^e[u, I]) = [u^{p^e}, I^{[p^e]})$. Since $[u, I] \in N_1$, we have a relation

$$[u, I] = \sum_{j=1}^{h} r_j [u^{Q_j}, I^{[Q_j]}]$$

where $1 < Q_1 < \cdots < Q_h$ are suitable powers of p. Since $0^*/0^+$ has finite length, we can choose q, a power of p, such that $m^q 0^* \subseteq 0^+$. Taking qth powers, we get:

$$[u^q, I^{[q]}] = \sum_{j=1}^{h} r_j^q [u^{qQ_j}, I^{[qQ_j]}].$$

The largest bracket exponent occurring on I here is qQ_h. This says that

$$x^{qQ_h - q} u^q \equiv \sum_{j=1}^{h} r_j^q x^{qQ_h - qQ_j} u^{qQ_j} \mod I^{[qQ_h]}.$$

For each term that such that r_j is not a unit, r_j^q multiplies 0^* into 0^+. Thus, if we work modulo $I^{[qQ_h]}R^+$, we may omit all terms on the right except those for which r_j is a unit. Therefore, after a change of notation, we have

$$x^{qQ_h - q} u^q \equiv \sum_{j=1}^{h} r_j^q x^{qQ_h - qQ_j} u^{qQ_j} \mod I^{[qQ_h]}R^+,$$

where every r_j^q occurring on the right is a unit. If there are no r_j that are units, then $x^{qQ_h - q} u^q \in I^{[qQ_h]}R^+$. Because the x_j form a regular sequence on R^+, we find that

$u^q \in I^{[q]}R^+$, and so $u \in IR^+$, a contradiction. Thus, there is no loss of generality in supposing that $h \geq 1$, and then r_h is a unit. But then

$$u^{qQ_h} \in (x^{qQ_h - q}u^q)R^+ + (x^{qQ_h - qQ_j}u^{qQ_j} : j < h)R^+ + I^{[qQ_h]}R^+.$$

The equational lemma developed in Step 7 then implies that $u \in IR^+$, giving a contradiction, and completing the proof, once we have finished Step 5 by showing that $0^*/0^+$ has finite length.

Step 7. Here, Q will play the role of qQ_h, and S the role of R^+. Of course, R^+ has no proper integral extension rings that are integral domains.

Equational Lemma. *Let S be a ring (respectively, a domain) of prime characteristic $p > 0$, and let $x_1, \ldots, x_d$ be nonzerodivisors in S. Let $x = x_1 \cdots x_d$. Let $p^e = q < Q$ be powers of p, and suppose that $u \in S$ is such that*

$$u^Q \in (x^{Q-1}u,\, x^{Q-p}u^p,\, x^{Q-p^2}u^{p^2},\, \ldots,\, x^{Q-q}u^q,\, x_1^Q,\, \ldots,\, x_d^Q)S.$$

Then there is a module-finite extension ring (respectively, domain) T of S such that $u \in (x_1, \ldots, x_d)T$.

Proof. The parenthetical statement about domains follows from the main statement, because if S is a domain we can choose T and then kill a prime ideal of T disjoint from $S - \{0\}$. We seek values $z_1, \ldots, z_d$ for indeterminates $Z_1, \ldots, Z_d$ in a module-finite extension of S such that $u = \sum_{k=1}^d z_k x_k$. We have that

$$(*) \qquad u^Q = \sum_{t=0}^{e} s_t x^{Q-p^t} u^{p^t} + \sum_{k=1}^{d} \sigma_k x_k^Q,$$

where the $s_t, \sigma_j \in S$. We first attempt to choose the Z_j so that they satisfy the equation that results when we substitute $\sum_{k=1}^d Z_k x_k$ for u in $(*)$. When we do this, we can expand each term in which $\sum_{k=1}^d Z_k x_k$ is raised to a power of the form p^a as a sum of d terms, indexed by k. The equation we want to hold will therefore hold if we can solve the d equations

$$(**_k) \qquad Z_k^Q x_k^Q = \sum_{t=0}^{e} s_t x^{Q-p^t} x_k^{p^t} Z_k^{p^t} + \sigma_k x_k^Q,$$

$1 \leq k \leq d$, since it is the sum of these. Each term in $(**_k)$ is formally divisible by the nonzerodivisor x_k^Q, and, factoring x_k^Q out in $(**_k)$ for each k, we see that if $y_k = \prod_{\nu \neq k} x_\nu$, so that $y_k x_k = x$, it suffices to solve the d equations:

$$Z_k^Q = \sum_{t=0}^{e} s_t y_k^{Q-p^t} Z_k^{p^t} + \sigma_k.$$

We can formally adjoin roots of these equations to S: the result is a module-finite free S-algebra T_0, and so the x_k are still nonzerodivisors. In T_0 we have an element $v = \sum_{k=1}^d z_k x_k$

in IT_0 that satisfies the same equation $(*)$ that u does. Let $w = u - v$. Subtracting the equations, we find that w satisfies $w^Q = \sum_{t=0}^{e} s_t x^{Q-p^t} w^{p^t}$. Dividing through by x^Q, we find that in $(T_0)_x$, we have $(w/x)^Q = \sum_{t=0}^{e} s_t (w/x)^{p^t}$. Since the x_k are nonzerodivisors, $T_0 \subseteq (T_0)_x$, and w/x is integral over T_0. Let $T = T_0[w/x]$, which is still module-finite over S. Then $w \in xT \subseteq IT$, and $v \in IT$, so that $u = w + v \in IT$, as required. $\square$

Thus, the entire proof will be complete if we can carry through Step 5, and show that $0^*/0^+$ has finite length if R is a minimum Krull dimension complete local Gorenstein domain counter-example for a full system of parameters.

Step 5. Now let R be a Gorenstein local ring and $c \in R$. Fix a system of parameters $x_1, \dots, x_d$. Let $I_t = (x_1^t, \dots, x_d^t)R$ as usual. We claim that c kills 0_M^* for every finitely generated R-module M if and only if c kills I_t^*/I_t for all I. The "only if" part is obvious. Suppose that c kills I_t^*/I_t for all t, that $u \in 0_M^*$ but that $cu \neq 0$. Choose h sufficiently large that $cu \notin m^h M$, and replace M by $M/m^h M$. Thus, we may assume that M has finite length. In fact, we may take a quotient of M by a maximal submodule not containing cu, and then cu will generate the socle in M. Thus, M can be taken to be a finite length essential extension of K, and will embed in E. But this means that M embeds in R/I_t for some t. This gives a contradiction.

But this means in the excellent Gorenstein domain case that the annihilator of 0^* in R is the test ideal of R. We shall denote this ideal by J_*, and we shall denote the annihilator of 0^+ by J_+. Taking duals in the short exact sequence $0 \to 0^+ \to 0^* \to 0^*/0^+ \to 0$ we get a short exact sequence: $0 \to (0^*/0^+)^\vee \to R/J_* \to R/J_+ \to 0$. Thus, $(0^*/0^+)^\vee \cong J_+/J_*$. Since dualizing does not affect whether a module has finite length, our problem is equivalent to showing that J_+/J_* has finite length. But this is a Noetherian module. Thus, it suffices show that it becomes 0 when we localize at prime ideal P strictly contained in m.

We shall do this in two steps. We shall first show that any element of J_+ becomes a test element in R_P, and then we shall show that every test element for R_P is, up to a unit, an element of $J_* R_P$. Let $u \in J_+$. Choose a system of parameters for R such that the first $h = \dim R_P$ elements are in P, so that the images of these are a system of parameters in R_P. It is harmless if we still denote these $x_1, \dots, x_d$. By the discussion in the first paragraph of Step 5, it will suffice to show that u multiplies $\big((x_1^t, \dots, x_h^t)R_P\big)^*_{R_P}$ into $(x_1^t, \dots x_h^t)R_P$ for all t. But in R_P, tight closure is the same as plus closure, which commutes with localization. Thus, it suffices to show that u multiplies $\big((x_1^t, \dots, x_h^t)R\big)^+$ into $(x_1^t, \dots, x_h^t)R$. Because u is in J_+, it multiplies $\big((x_1^t, \dots, x_h^t, x_{h+1}^N, \dots, x_d^N)R\big)^+$ into $(x_1^t, \dots, x_h^t, x_{h+1}^N, \dots, x_d^N)R$ for all N, and so u multiplies $\big((x_1^t, \dots, x_h^t)R\big)^+$ into $(x_1^t, \dots, x_h^t, x_{h+1}^N, \dots, x_d^N)R$ for all N. But the intersection of the latter over N is $(x_1^t, \dots, x_h^t)R$, as required, since every ideal of a local ring is closed in the m-adic topology.

Finally, we are left with the problem of showing that any test element for R_P is in $J_* R_P$.

Call an ideal J of the excellent local Gorenstein domain R an *F-ideal* if its annihilator in $H^d_m(R)$ is F-stable. This means that for all systems of parameters $x_1, \dots, x_d$ of R, $Ju \subseteq (x_1, \dots, x_d)R$ implies that $Ju^p \subseteq (x_1, \dots, x_d)^{[p]}R$. It is sufficient to check this for all the $x_1^t, \dots, x_d^t$ for one system of parameters $x_1, \dots, x_d$ as t varies.

Lemma (K. E. Smith). *Let R be an excellent Gorenstein local domain of characteristic $p > 0$. For every prime ideal P of R, if J is an F-ideal of R, then JR_P is an F-ideal of R_P.*

Proof. Choose a system of parameters $x_1, \dots, x_h, x_{h+1}, \dots, x_d$ for R such that the first h terms $x_1, \dots, x_h$ form a system of parameters for R_P. Now suppose we have that $Ju \subseteq (x_1^t, \dots, x_h^t)R_P$. We may assume that $u \in R$, and after replacing u by cu with $c \in R - P$, we may assume that $Ju \subseteq (x_1^t, \dots, x_h^t)R$. Then

$$Ju \subseteq (x_1^t, \dots, x_h^t, x_{h+1}^N, \dots, x_d^N)R$$

for all n, and so

$$Ju^p \subseteq (x_1^{pt}, \dots, x_h^{pt}, x_{h+1}^{pN}, \dots, x_d^{pN})R.$$

Taking the intersection as N varies, we get that $Ju^p \subseteq (x_1^{pt}, \dots, x_h^{pt})R$, as required. $\square$

The following result now completes Step 5, and also the entire proof that plus closure agrees with tight closure for locally excellent domains of characteristic $p > 0$.

Theorem (K. E. Smith). *Let R be a complete local Gorenstein domain of prime characteristic $p > 0$. Let J_* denote the test ideal of R. Then for every prime ideal P of R, J_*R_P is the test ideal for R_P.*

(Note: for the proof of Step 5, we only need the harder half of this, which asserts that that every test element for R_P is in J_*R_P.)

Proof. First suppose that c is a test element of R. Choose part of a system of parameters $x_1, \dots, x_h$ for R such that $x_1, \dots, x_h$ is a system of parameters in R_P. It suffices to show that $c\big((x_1^t, \dots, x_h^t)R_P\big)^* \subseteq (x_1^t, \dots, x_h^t)R_P$ for all t. But we showed last semester that tight closure commutes with localization for regular sequences, so that $\big((x_1^t, \dots, x_h^t)R_P\big)^* = \Big(\big((x_1^t, \dots, x_h^t)R\big)^*_R\Big)R_P$, and the result follows because c, a test element of R, multiplies $\big((x_1^t, \dots, x_h^t)R\big)^*_R$ into $(x_1^t, \dots, x_h^t)R$.

We deduce the other inclusion from the Lemma above on localization of F-ideals. We observed earlier that J_* is the annihilator of 0^*. Since R is complete, this means that $0^* = \operatorname{Ann} J_*$. It follows that J_* is an F-ideal. By the Lemma, $(J_*)_P$ is an F-ideal. Note that every complete local domain has a nonzero test element, and so neither J_* nor $(J_*)_P$ is zero. Let N be the F-stable annihilator of $(J_*)_P$ in $H = H_{PR_P}(R_P)$. Then $N \subseteq 0^*_H$, because if $d \in (J_*)R_P - \{0\}$ and $v \in N$, then $v^q \in N$ for all q, and so $dv^q = 0$ for all N. Any test element for R_P therefore kills N. Over $\widehat{R_P}$, N is the annihilator of $(J_*R_P)\widehat{R_P}$, and so any element of R_P that kills N is in $(J_*R_P)\widehat{R_P} \cap R_P = J_*R_P$. Thus, every test element for R_P is in J_*R_P, as required. $\square$