715: Lectures of October 3, 2002

Let R be a Noetherian F-finite domain of characteristic $p > 0$. We next want to prove that the strongly F-regular locus is Zariski open, and that if R_c is strongly F-regular then, if $c \in R^\circ$, c has a power that is a test element.

Before doing this, we want to make a remark about the test elements in a domain: taken together with 0, they form an ideal. In fact, this ideal is the same as $\bigcap_{N \subseteq M} N :_R N^*_M$, as $N \subseteq M$ run through finitely generated modules. We shall call this ideal the *test ideal* for R and denote it $\tau(R)$. Quite generally, the test elements in R are the same as the elements of $\tau(R) - R^\circ$.

Since the pair $N \subseteq M$ may be replaced by the pair $0 \subseteq M/N$, it suffices to consider $\bigcap_M \mathrm{Ann}_R\, 0^*_M$ as M runs through finitely generated modules. In fact, it suffices to consider modules of finite length with a one-dimensional socle — on the face of it, there might be more in the intersection when we restrict attention to these, but if $u \in 0^*_M$ and is not killed by $r \in R$, we can choose $N \subseteq M$ maximal with respect to not containing ru, and then M/N will be killed by a power of some maximal ideal m (so that it can be thought of as a module over R_m), will have a one-dimensional socle and so embed in E, the injective hull of the residue field of R_m. Note that the image v of u in M/N will be in the tight closure of 0 in M/N and we will still have $rv \neq 0$. Thus, $\tau(R) = \bigcap_M \mathrm{Ann}_R 0^*M$ as M runs through finite length submodules of injective hulls of the residue fields of the various local rings R_m for m maximal in R. Note that for each m we need only include a countable increasing family of submodules of E whose union is E (e.g., the family $\mathrm{Ann}_E m^t$ at t varies), because every finite length submodule will be contained in a member of the family, and the larger submodule has smaller annihilator.

It follows that if the local rings of R at maximal ideals are approximately Gorenstein, e.g., if R is reduced an locally excellent (thus, we are imposing a rather weak condition), then $\tau(R) = \bigcap_I I :_R I^*$ as I runs through the ideals of R. In fact, instead of using all ideals, it suffices, for each maximal ideal m of R, to choose a decreasing sequence of m-primary irreducible ideals cofinal with the powers of m, and to intersect all of the ideals $I :_R I^*$ as I runs through the union of all these sequences. Thus, under mild conditions, it does not matter whether one uses only ideals or all finitely generated modules in defining the test ideal $\tau(R)$.

We now return to the tasks discussed in the first paragraph. We need some lemmas in order to prove the results that we want.

Lemma. *If R as above is such that R_c is strongly F-regular, then for every $d \in R - \{0\}$ there are powers of p, q and Q, and an R-linear map $R^{1/q} \to R$ sending $d^{1/q}$ to c^Q. Any power of p that is $\geq Q$ may also be used.*

Proof. . By hypothesis, there exists q and an R_c-linear map $(R^{1/q})_c \to R_c$ that sends $d^{1/q}$ to 1. Restrict this to a map α on $R^{1/q}$, which is a finitely generated R-module. The image of each generator can be written in the form r/c^h, and if N is the maximum of the values of h needed, then $c^N\alpha$ maps $R^{1/q}$ into R and has value c^N on $d^{1/q}$. Clearly, N may be replaced by any larger integer, and so may be assumed to be a power Q of p, and it is clear that any larger power of p may also be used. □

Theorem. *Let R be a Noetherian F-finite domain of characteristic $p > 0$. Suppose $c \neq 0$ is such that R_c is F-regular. Then R is F-regular iff the map $R \to R^{1/Q}$ sending 1 to $c^{1/Q}$ splits for some Q.*

Proof. First recall that the hypothesis implies that the map $R \subseteq R^{1/Q}$ splits as well, and, by taking q' th roots, we get a splitting of $R^{1/q'} \subseteq R^{1/q'Q}$ also.

Let $d \neq 0$ be given. By the preceding Lemma, we have an R-linear map $R^{1/q} \to R$ mapping $d^{1/q}$ to c^Q, where $Q \gg 0$. Choose q' so large that the map $R \to R^{1/q'}$ sending 1 to $c^{1/q'}$ splits. Taking $q'Q$ th roots, we have a map $R^{1/qq'Q} \to R^{1/q'Q}$ sending $d^{1/qQ}$ to $c^{1/q'}$ that is R-linear. There is an R-linear map $R^{1/q'Q} \to R^{1/q'}$ that sends $c^{1/q'}$ to itself, by the assertion of the first paragraph. Finally, compose with the map $R^{1/q'} \to R$ that sends $c^{1/q'}$ to 1 to get a map $R^{1/qq'Q} \to R$ that sends $d^{1/qq'Q}$ to 1. □

By a theorem of Kunz, the flatness of Frobenius implies that R is regular. Thus, if $c \neq 0$ in R is such that $(R^{1/p})_c$ is R_c-free, then R is regular. We shall not prove this here, but we do want to show R_c is F-regular without using this. We need some preliminaries.

First, note that if M is any finitely generated R-module, where R is a Noetherian domain, and $u_1, \ldots, u_h$ is a maximal set of elements of M that are linearly independent over R, then we have an embedding $G \subseteq M$, where G is the R-free module spanned by the u_j, and that the cokernel M/G is a torsion module over R. (If $u_{h+1} \in M$ represented an element with trivial annihilator in R, then $u_1, \ldots, u_{h+1}$ would be a larger linearly independent set of elements in M.) If $c \neq 0$ is an element that annihilates M/G, then $G_c \to M_c$ is an isomorphism, i.e., M_c is R_c-free.

Note also that if $A \to B \to C$ is such that B is a free algebra over A with basis $\{b_\lambda : \lambda\}$ and if C is a free B-algebra with basis $\{c_\mu : \mu\}$ then C is a free A-algebra with basis $\{b_\lambda c_\mu : \lambda, \mu\}$. If $R^{1/p}$ is free over R then taking q th roots shows that $R^{1/pq}$ is free over $R^{1/q}$ for all q, and iterated use of the fact in the preceding sentence shows that $R^{1/q}$ is free over R for all q.

Lemma. *With R a Noetherian domain that is F-finite of positive characteristic p, we may choose $c \neq 0$ such that $R_c^{1/p}$ is R-free, and, for such a c, R_c is strongly F-regular. That is, if $R^{1/p}$ is R-free, then R is strongly F-regular.*

Proof. The fact that we can choose c is immediate from the first of the two paragraphs preceding the statement of the Lemma, with $M = R^{1/p}$. The hypothesis is preserved by localizing at a maximal ideal m of R. Since $R_m^{1/p} \cong (R_m)^{1/p}$, and since a ring is strongly F-regular if all of its local rings at maximal ideals are, we may assume without loss of generality that (R, m) is local. By the remark in the preceding paragraph, $R^{1/q}$ is R-free for all q. Now let $d \in R - \{0\}$ be given. The intersection of the powers of m is zero, and $m^{[q]} \subseteq m^q$. Thus, we can choose $q \gg 0$ such that $d \notin m^{[q]}$. Taking q th roots, we find that $d^{1/q} \notin mR^{1/q}$. By Nakayama's lemma, $d^{1/q}$ is part of a minimal basis for $R^{1/q}$ over R, and since $R^{1/q}$ is R-free, this means that it is part of a free basis for $R^{1/q}$ over R. But then an R-linear map $R^{1/q} \to R$ may be specified arbitrarily on the free basis, so that there is an R-linear map $R^{1/q} \to R$ sending $d^{1/q} \to 1$ and the other basis elements to, say, 0. (What it does to the other elements of the free basis does not matter.) Thus, R is strongly F-regular. □

Of course, when $R^{1/p}$ is R-free (or even R-flat) R is regular by the stronger result of Kunz mentioned earlier.

Theorem. *Let R be a Noetherian domain that is F-finite of positive characteristic p. Then the strongly F-regular locus $\{P \in \operatorname{Spec} R : R_P$ is strongly F-regular$\}$ is open.*

Proof. Choose an element $c \in R - \{0\}$ such that $(R^{1/p})_c$ is R-free, so that R_c is weakly F-regular. Now suppose that R_P is strongly F-regular. Then we can choose q so that the map $R_P \to (R^{1/q})_P$ that sends 1 to $c^{1/q}$ splits. The restriction of the splitting to $R^{1/q}$ takes values in $R \cdot \frac{1}{a}$ for some element $a \notin P$, and so we get a splitting of the map $R_a \to (R_a)^{1/q}$ that sends 1 to $c^{1/q}$. But then we may localize further to get a splitting of the corresponding map for R_Q for every prime Q that does not contain a, which shows that one has strong F-regularity locally on a Zariski neighborhood of P. □

An element of R° is called a *locally stable* (respectively *completely stable*) *test element* if it is test element in every local ring of R (respectively, in the completion of every local ring of R). It is easy to see that a locally stable test element is a test element for every localization of R, including R itself, and that a completely stable test element is locally stable (a test element for the completion of a local ring R that is in R is a test element for R). We shall soon see that the theory of strongly F-regular rings provides a large source of completely stable test elements.

Theorem. *With R as above, if $c' \in R$ is such that $R_{c'}$ is strongly F-regular, then c' has a power that is a test element. Moreover, one can choose a power c of c' such that there is an R-linear map $R^{1/p} \to R$ sending 1 to c, and then c^3 is a test element.*

Proof. Since $R_{c'}$ is strongly F-regular, there is an $R_{c'}$-linear map from $R_{c'}^{1/p} \to R_{c'}$ sending 1 to 1. We can restrict it to $R^{1/p}$ and clear denominators to get an R-linear map, say h, from $R^{1/p} \to R$ sending 1 to a power c of c'. We next claim that there is a map from $R^{1/q} \to R$ sending 1 to c^2 for all q. If $q = p$ we may simply use ch. We use induction. Assuming such a map from $R^{1/q} \to R$ we take p th roots to get a map $R^{1/pq} \to R^{1/p}$ that is $R^{1/p}$-linear and sends 1 to $c^{2/p}$. Multiplying this map by $c^{(p-2)/p}$ we get a map that sends $R^{1/pq} \to R^{1/p}$ such that the image of 1 is c. Composition with h produces a map that sends 1 to c^2, as required.

Now suppose that $u \in I^*$. We want to show that $c^3 u \in I$. We have $d \in R - \{0\}$ such that $du^q \in I^{[q]}$ for all $q \gg 0$, and so $d^{1/q} u \in IR^{1/q}$ for all $q \gg 0$. For sufficiently large q, we can map $R^{1/q'}$ to R via an R-linear map such that $d^{1/q'}$ maps to c^Q. Taking Q th roots we have a map $R^{1/q'Q} \to R^{1/Q}$, linear over the latter, that sends $d^{1/q'Q}$ to $c \in R^{1/Q}$. Now $d^{1/q'Q}u \in IR^{1/q'Q}$, and so, applying the map, we have that $cu \in IR^{1/Q}$. But there is a map $R^{1/Q} \to R$ such that 1 maps to c^2, so that $c^3u \in I$. $\square$

Excellent rings. Noetherian rings that come up in algebraic geometry, number theory, several complex variables, and so forth have better properties than arbitrary Noetherian rings. This idea was formalized by A. Grothendieck in his *Éléments de géométrie algébrique*, where the notion of an *excellent ring* was introduced. It is beyond the scope of this course to do a detailed treatment of the notion, but it is often the right hypothesis for our theorems. We therefore give the definition and some properties that we shall need. For further details, we refer the reader to H. Matsumura's book *Commutative Algebra*, W.A. Benjamin, New York, 1970, which has a self-contained treatment of the subject that avoids excessive detail and generality. We recall that a Noetherian ring is called *catenary* or *catenarian* if for any two prime ideals $P \subseteq Q$, all saturated chains of primes joining P to Q have the same length (where *saturated* means that for any two consecutive primes in the chain, there are no primes strictly between them).

Second, we recall the very useful notion of *fiber* of a ring homomorphism. Let $R \to S$ be a homomorphism of Noetherian rings and P a prime ideal of R. Let κ_P denote the field R_P/PR_P, which is canonically isomorphic with the fraction field of R/P. The elements of $\mathrm{Spec}\,(\kappa_P \otimes_R S)$ are in bijective correspondence with the primes of S lying over P, that is with the set-theoretic fiber $f^{-1}(P)$ of the induced map $f : \mathrm{Spec}\,S \to \mathrm{Spec}\,R$, and the ring $\kappa_P \otimes_R S$ is referred to as the *fiber* of the map $R \to S$. (One might use the term *scheme-theoretic fiber* also.) Note that the condition for a prime Q of S to lie over P is that it contain the image of P, and hence contain PS (such primes correspond to the primes of S/PS) and be disjoint from the image of $W = R - P$, and such primes correspond to those of $W^{-1}S = S_P$. The two conditions together yield precisely primes that correspond to those of the ring $S_P/PS_P \cong (R_P/PR_P) \otimes_R S = \kappa_P \otimes S$. The fiber over P is called *regular* if $\kappa_P \otimes_R S$ is a regular ring. It is called *geometrically regular* if the fiber is a *geometrically regular algebra* over κ_P: this means that for every finite algebraic extension λ of κ_P, $\lambda \otimes_{\kappa_P} (\kappa_P \otimes_R S)$, which is $\cong \lambda \otimes_R S$, is regular as well. The condition is automatic for separable base changes, and so one need only impose it when λ is a finite purely inseparable extension of κ_P.

If R is a domain with fraction field L, the fiber $L \otimes_R S$ of $R \to S$ over the prime ideal (0) in R is called the *generic fiber*. If (R, m) is local the fiber S/mS of $R \to S$ over m is called the *closed fiber*.

The *formal fibers* of a local ring R are the fibers of the map $R \to \widehat{R}$. A ring is called *excellent* if it is Noetherian, universally catenary, the formal fibers of all of its local rings are geometrically regular, and the regular locus is open in $\mathrm{Spec}\,S$ for every finitely generated R-algebra S (much weaker but more technical statements can be substituted for this last condition).

This is a very technical definition, but the main point for us is that good properties of local rings of R are preserved when one completes. Here is a summary of some good properties of excellent rings:

If R is excellent, so is every homomorphic image and localization of R, and every finitely generated R-algebra. Complete local rings (hence, fields) are excellent. The integers is excellent. Analytic local rings are excellent (these are homomorphic images of rings of convergent power series over $\mathbb{C}$). By a theorem of Kunz, F-finite rings are excellent.

If R is an excellent domain, the integral closure of R is a finitely generated R-module. If R is excellent and local, then if R is reduced, so is $\widehat{R}$. Also, if R is normal, so is $\widehat{R}$. (However, when R is a domain and not necessarily normal, $\widehat{R}$ need not be a domain. E.g., let $A = \mathbb{C}[x, y]$, and let T by its localization at $(x,\, y)$. Let $R = T/(y^2 - x^2 - x^3)$. This is a domain: the polynomial $x^2 + x^3$ has no square root in $\mathbb{C}(x)$. However, if one completes, $\widehat{R} = \widehat{T}/(y^2 - x^2 - x^3) = \mathbb{C}[[x,\, y]]/(y^2 - x^2 - x^3)$ is no longer a domain. The polynomial now factors into two irreducibles, corresponding to the two square roots of $x^2 + x^3$ in $\mathbb{C}[[x]]$, $\pm x(1+x)^{1/2}$. Note that $(1+x)^{1/2}$ is in the power series ring $\mathbb{C}[[x]]$: one can find the power series square root explicitly using the binomial theorem.)

Experience has shown that when a map $R \to S$ is flat and both R and all fibers have a certain good property, S tends to have that same good property. One important instance is this:

Proposition. *IF $R \to S$ is a flat homomorphism of Noetherian rings such that R is regular and all the fibers are regular, then S is regular.*

Proof. The result is immediate from part (b) of the Lemma just below, since the issue is local on S (and if one localizes S at prime, one may also localize R at the contraction of that prime. $\square$

Lemma. *Let $(R,\, m) \to (S,\, n)$ be a local map (i.e., the image of m is contained in n) of local Noetherian rings that is flat.*
(a) *Then $\dim S = \dim R + \dim S/mS$.*
(b) *If R and S/mR are regular, then S is regular.*

Proof. For part (a), note that if $\dim R = 0$ then m is nilpotent and killing mS, which is also nilpotent, will not change the dimension of S. The result follows. More generally, we can kill the ideal of all nilpotents of R and its expansion to S. Thus, we may assume that R is reduced and of positive dimension. Then we can choose an element x of R not in any minimal prime, and x will be a nonzerodivisor in R. By the flatness of S over R, x is also a nonzerodivisor in S. The result now follows by induction applied to the flat local map $R/xR \to S/xS$. Killing a nonzerodivisor in the maximal ideal of a local ring decreases the dimension by 1. Note that killing $I \subseteq R$ and $IS \subseteq S$ gives a flat map $R/I \to S/IS$ without affecting the fiber over the maximal ideal.

For part (b), note that m is generated by $\dim R$ elements, and since S/mS is generated by $\dim\,(S/mS)$ is elements, we get that n is generated by at most $\dim R + \dim\,(S/mS) = \dim S$ elements, while we know that at least $\dim S$ elements are required. Thus, S is regular. $\square$

We can now see that there are completely stable test elements in an F-finite domain. In the course of the proof we need that when R is an F-finite local domain, the obvious map $A = \widehat{R} \otimes R^{1/q} \to \widehat{R}^{1/q} = B$ is an isomorphism (we only need the case where $q = p$

just below, but we shall soon need the general case). We leave this an exercise, but here are some suggestions about doing it. We are dealing with excellent local rings, so that completions of reduced rings are reduced. The ring A is the same as the completion of $R^{1/q}$, since $R^{1/q}$ is module-finite over R. To prove injectivity, use that if an element is in the kernel, its q th power is in the kernel. To prove surjectivity, note that any element b of B is the q th root of the limit of a Cauchy sequence of elements of R. But q th roots of the elements in the Cauchy sequence exist in A. Check that those q th roots form a Cauchy sequence in A whose limit will map to b.

Theorem. *Suppose that R is a Noetherian F-finite domain of positive characteristic p. Let c' be an element of R such that $R_{c'}$ is regular. Let c be a power of c' such that there is an R-linear map $R^{1/p} \to R$ sending 1 to c. (We proved earlier that there is such a power.) Then c^3 is a completely stable test element. In particular, every F-finite domain has a completely stable test element.*

Proof. The hypotheses are stable under passage to a local ring of R, and so c is locally stable by the theorem we proved earlier. Thus, we may assume that R is local. To show that c^3 is completely stable, we need to prove that the hypotheses pass to the completion. One point is that $\widehat{R}_c$ is regular. The reason for this is that since $R \to \widehat{R}$ is flat, this remains true when we localize at c, while the fibers of $R_c \to \widehat{R}_c$ are a subset of the fibers of $R \to \widehat{R}$ (corresponding to those primes of R that do not contain c). Thus, since R is F-finite and, therefore, excellent, it follows that the fibers are regular, and so $\widehat{R}_c$ is regular. We get the required map $\widehat{R}^{1/p} \to \widehat{R}$ sending 1 to c by completing, which is the same as tensoring over R with $\widehat{R}$. One needs to verify that the obvious map $\widehat{R} \otimes R^{1/p} \to \widehat{R}^{1/p}$ is an isomorphism, which was discussed just above $\square$

Corollary. *If R is a local and F-finite domain, R is strongly F-regular if and only if $\widehat{R}$ is strongly F-regular.*

Proof. Fix c such that R_c is regular (and, hence, $\widehat{R}_c$ is regular). Whether there is a splitting of the map $\phi : R \to R^{1/q}$ that sends 1 to $c^{1/q}$ is independent of whether we complete or not. Thus, if either ϕ or $\widehat{\phi} : \widehat{R} \to \widehat{R}^{1/q}$ splits for a certain q, so does the other. But one has strong F-regularity for R iff ϕ splits for large q, and the same holds for $\widehat{R}$ and $\widehat{\phi}$. $\square$

Lemma. *Let R be a local ring with a completely stable test element c. The R is weakly F-regular if and only if $\widehat{R}$ is weakly F-regular.*

Proof. A sufficient condition for weak F-regularity is that 0 be tightly closed in every finite length R-module. These modules are "the same" whether we work over R or $\widehat{R}$ (those that are killed by m^N are the same as the modules over $R/m^N \cong \widehat{R}/m^N\widehat{R}$), and for such a module M, $F^e(M)$ over R may be identified with $F^e(M)$ over $\widehat{R}$. Because one has an element of R that is a test element in $\widehat{R}$, checking whether an element of M is in the tight closure of 0 is independent of which ring one is working over. $\square$