715: Lectures of October 10, 2002

Criteria for normality. A Noetherian ring is normal if and only if principal ideals are unmixed and the localization at any height one prime is a DVR. An equivalent statement is that the localization at any depth one prime is a DVR (equivalently, regular). If the singular locus is closed and defined by I, this is equivalent to the condition that I have depth at least 2, i.e., that I be the unit ideal or contain a regular sequence of length at least 2. (Suppose that I has depth 2. If a principal ideal has an associated prime of height 2, the localization at that prime has dimension at least two but depth only one. Therefore, the prime is not regular — but that means it contains the depth two ideal defining the non-regular locus. Likewise, if the localization of R at height one prime is not a DVR, then that height one prime is singular, and this will force depth two for the localization, a contradiction. Therefore, R is normal. On the other hand, suppose that R is normal but the defining ideal of the singular locus has depth at most 1. Localizing at a suitable associated prime of I produces a non-regular local ring of depth at most 1. The dimension cannot be one, since this ring is normal. But a normal ring of dimension two or more has depth at least two, because the associated primes of a principal ideals have height one and can be avoided.

Proposition. *If S is flat over R with regular fibers and R is normal, then S is normal. Hence, if R is normal excellent local, $\widehat{R}$ is normal.*

Proof. It suffices to show that if Q is a prime ideal of S of depth one then S_Q is regular. Suppose that Q lies over P in R. Then $R_P \to S_Q$ is faithfully flat with a regular closed fiber. It follows that the depth of R_P is at most one, and so R_P is regular. But then S_Q is regular.

If R is excellent local and normal, we know that $R \to \widehat{R}$ has regular fibers, and it follows that $\widehat{R}$ is normal. $\square$

Exercise If $R \to S$ is flat with reduced fibers, and R is reduced, then S is reduced. (Suggestion: localize at the multiplicative system of nonzerodivisors in R. Then R becomes a finite product of fields. In this case, S is a finite product of field extensions, each of which is a fiber of the original map.) As a corollary, if R is excellent local reduced, then $\widehat{R}$ is reduced.

Proposition. *If R is an excellent, reduced local ring such that for every minimal prime P, $dim\, R/P = dim\, R$, this property is preserved when we complete.*

Proof. We first show that if R is an excellent local domain, then all minimal primes of $\widehat{R}$ have the same height. Consider the normalization of R, which is module-finite over R and is a semilocal ring S. Then the completion of S with respect to the maximal ideal of R is the product of the completions of the various localizations of S at its maximal ideals with respect to those maximal ideals. Therefore the completion of R embeds in a finite product of completed normal local rings, each of which has dimension equal to that of R and is normal (and consequently a domain). (This also shows that the number of minimal primes of R is at most the number of maximal ideals of the normalization of R.)

More generally, if R is reduced and equidimensional, R embeds in the product of the domains R/P obtained by killing the various finitely many minimal primes of R. Its completion embeds in the product of their completions, from which the result follows. $\square$

We next want to discuss operations like sum, product, colon, and intersection on ideals generated by monomials (with non-negative coefficients) in elements from a permutable regular sequence in the ambient ring R. It will be convenient to introduce a base ring Λ that maps to R, which we may take to be $\mathbb{Z}$. If R contains a field K, one may also take Λ to be K. This will make no difference in the discussion which follows.

Let $T = \Lambda[X_1, \ldots, X_n]$, where X_i are indeterminates, and let $x_1, \ldots, x_n$ be a permutable regular sequence in R. Let $\mathcal{M}$ (resp., $\mathcal{N}$) denote the set of all monomials (including 1) in the X_i (resp., the x_i). If I is an ideal generated by monomials in the X_i in T, then IR is a monomial ideal in the x_i in R, and all monomial ideals in the x_i arise by expansion in this way. Note that each monomial ideal in T is spanned over Λ by a subset of the free basis $\mathcal{M}$. The sum of two monomial ideals I_1, I_2 in T is spanned over Λ by $(I_1 \cap \mathcal{M}) \cup (I_2 \cap M)$, and it it easy to see that their intersection is spanned over Λ by $(I_1 \cap \mathcal{M}) \cap (I_2 \cap \mathcal{M})$. This implies that for monomial ideals in T, sums and intersections can be computed purely formally, and intersection distributes over sum. Thus, given $\mu_1, \ldots, \mu_h \in \mathcal{M}$ and $\nu_1, \ldots, \nu_k \in \mathcal{M}$,

$$(\mu_1, \ldots, \mu_h) \cap (\nu_1, \ldots, \nu_k) = (\sum_{i=1}^{h} \mu_i T) \cap (\sum_{j=1}^{k} \nu_j T) = \sum_{i,j} \big((\mu_i) \cap (\nu_j)\big) = \sum_{i,j} \mathrm{LCM}\,(\mu_i, \nu_j) T,$$

and this is $\big(\mathrm{LCM}\,(\mu_i, \nu_j) : i, j\big)T$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, let $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Then it is easy to see that $\mathrm{LCM}\,(X^\alpha, X^\beta) = (X^\gamma)$, where for each i, $\gamma_i = \max\{\alpha_i, \beta_i\}$.

For elements $a, b \in \mathbb{N}$ let $a \dot{-} b = \max\{a - b, 0\}$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, let $\alpha \dot{-} \beta = (\alpha_1 \dot{-} \beta_1, \ldots, \alpha_n \dot{-} \beta_n)$. We can now compute $I_1 :_T I_2$ for monomial ideals I_1 and I_2 of T quite formally. First note that in total generality, in any ring R, $I :_R (r_1, \ldots, r_k) = \bigcup_{i=1}^{k} (I :_R (r_i)$. Thus, it suffices to consider the calculation of $(_1, \ldots, (_\mu h) : T\nu$ when the μ_i and ν are in $\mathcal{M}$.

Summary of Tor. We recall some basic facts about Tor over a commutative ring R. For each i, $\mathrm{Tor}_i^R(A, B)$ is a functor from pairs of R-modules to R-modules which is covariant in each of the variable modules A, B if the other is fixed. To compute Tor, we may take a free (or projective, or even flat) left resolution of A, say $\cdots \to G_i \to \cdots G_1 \to G_0 \to A \to 0$. Let $G_\bullet$ denote the complex $\cdots \to G_i \to \cdots G_1 \to G_0 \to 0$ with the augmentation omitted, so that $H_0(G_\bullet) = A$. Then $\mathrm{Tor}_i^R(A, B) \cong H_i(G_\bullet \otimes_R B)$. $Tor_0^R(A, B)$ may be identified canonically, as a functor of two variables, with $A \otimes_R B$. We frequently omit the superscript ^{R} when the ring is clear from the context. Tor commutes with arbitrary direct sums and direct limits. $\mathrm{Tor}_i(A, B) \cong \mathrm{Tor}_i(B, A)$, and vanishes if $i \geq 1$ provided that either A or B

is flat. There is a functorial long exact sequence for Tor: if $0 \to A \to B \to C \to 0$ is a short exact sequence and D is any R-module one gets:

$$\cdots \to \mathrm{Tor}_i(A, D) \to \mathrm{Tor}_i(B, D) \to \mathrm{Tor}_i(C, D) \to \mathrm{Tor}_{i-1}(A, D) \to \cdots$$

$$\to \mathrm{Tor}_1(C, D) \to A \otimes D \to B \otimes D \to C \otimes D \to 0.$$

If R is Noetherian and A, B are finitely generated modules, then all of the $\mathrm{Tor}_i^R(A, B)$ are finitely generated: one may use a free resolution of A by finitely generated free modules to compute them.

If A is any module, G is free, or projective, or flat, and $0 \to B \to G \to A \to 0$ is exact (when G is free, B is called a *(first) module of syzygies* of A, then the long exact sequence for Tor shows that $\mathrm{Tor}_1(A, M) \cong \mathrm{Ker}\,(B \otimes M \to G \otimes M)$ while $\mathrm{Tor}_i(B, M) \cong \mathrm{Tor}_{i+1}(A, M)$ for all $i \geq 1$. $\mathrm{Tor}_i(A, B)$ is killed by both the annihilator of B and the annihilator of A for all $i \geq 0$.

Exercises. Let T be a ring and M a T-module. Let I be an ideal of T.
(a) $\mathrm{Tor}_1^T(T/I, M) = 0$ if and only if the obvious map $I \otimes_T M \to M$ is injective, giving an isomorphism of $I \otimes_T M$ with IM.
(b) If $\mathrm{Tor}_i^T(T/I, M) = 0$ for all $i \geq 1$, then $\mathrm{Tor}_i^T(I,\, M) = 0$ for all $i \geq 1$.
(c) Let A and M be T-modules such $\mathrm{Tor}_j^T(A,\, M) = 0$ for all $j \geq 1$. Let $z \in T$ be a nonzerodivisor in A, and a nonzerodivisor on $A \otimes_T M$. Prove that $\mathrm{Tor}_j^T(A/zA,\, M) = 0$ for all $j \geq 1$. [Suggestion: apply the long exact sequence for Tor obtain from the short exact sequence $0 \to A \xrightarrow{z} A \to A/zA \to 0$ by tensoring with M over T. Note that the case $j = 1$ is different from the case $j > 1$.]
(d) Recall the a sequence of elements $z_1, \ldots, z_n$ in a ring T is called a *possibly improper regular sequence* on the T-module M z_{i+1} is not a zerodivisor on $(z_1, \ldots, z_i)M$ for every i, $0 \leq i < n$. It is called a *regular sequence on* M if, in addition, $(z_1, \ldots, z_n)M \neq M$. Note that if $M = R$ is a T-algebra, then the z_i form a (possibly improper) regular sequence on M if and only if their images form such a sequence in the ring R. Prove that if $z_1, \ldots, z_n$ form a possibly improper regular sequence on M, then $\mathrm{Tor}_j^T(T/(z_1, \ldots, z_n),\, M) = 0$ for all $j \geq 1$. [Suggestion. Use induction on n: the case $n = 0$ is obvious. The inductive step follows from part (c) with $A = T/(z_1, \ldots, z_n)T$ and $z = z_{n+1}$.]

Theorem. *Let $T = \Lambda[X_1, \ldots, X_n]$, where Λ is either $\mathbb{Z}$ or else a subfield of R, and let $x_1, \ldots, x_n$ be a sequence of elements of R which forms a (possibly improper) regular sequence in every order. Extend the obvious map of Λ to R to all of T by sending every X_i to the corresponding x_i, and let $\mathcal{M}$ be the set of all monomials in the X_i. Then*
(a) *For any two ideals I_1, I_2 generated by elements of $\mathcal{M}$, we have that $I_1R \cap I_2R = (I_1 \cap I_2)R$, and $I_1R :_R I_2R = (I_1 :_T I_2)R$. (It is also true that $I_1R + I_2R = (I_1 + I_2)R$, and that $(I_1R)(I_2R) = (I_1I_2)R$, but these statements are trivial and hold for all ideals of R, not just those generated by monomials in a regular sequence).*
(b) *For every ideal I generated by elements of $\mathcal{M}$, $\mathrm{Tor}_i^T(T/I,\, R) = 0$.*

Proof. We prove part (b) first. By Noetherian induction we may reduce to the case of an ideal maximal among those generated by elements of $\mathcal{M}$ for which the theorem fails. We shall show that such an ideal I is generated by powers of the various X_i. If not, there will be some minimal generator $\mu = X_{i_1}^{a_1} \cdots X_{i_h}^{a_h}$ where $h \geq 2$, all the i_j are distinct, and all the a_i are positive, i.e., a minimal generator that involves at least two of the X 's Let μ_j denote the monomial μ/X_{i_j} in which the exponent on X_{i_j} is decreased by one and the exponents in μ remain the same. Then $(\mu) = \bigcap_j (\mu_j)$. The μ_j cannot belong to I, or else the minimality of generator μ would be contradicted. Thus, the ideals $I + (\mu_j)$ are all strictly larger than I, and I is the intersection of them, because intersection distributes over sum for monomial ideals. Since I is an intersection of finitely many strictly larger monomial ideals, it can be written as the intersection of two such ideals, say $I = I_1 \cap I_2$. Then $\mathrm{Tor}_j^T(T/J, R)$ vanishes for $J = I_1$, for $J = I_2$, and for $J = I_1 + I_2$, by the maximality of I as a counterexample. There is a short exact sequence $0 \to T/(I_1 \cap I_2) \to T/I_1 \oplus T/I_2 \to T/(I_1 + I_2) \to 0$ where the map on the left sends $t + (I_1 \cap I_2)$ to $(t + I_1) \oplus (t + I_2)$ and the map on the right sends $(t_1 + I_1) \oplus (t_2 + I_2)$ to $(t_1 - t_2) \oplus (I_1 + I_2)$. Since Tor commutes with direct sum, the $Tor_j(\_, R)$ vanish for $j \geq 1$ when the first variable is either the middle term of the right term, and the long exact sequence for Tor implies that it vanishes for the left term as well, as required. This proves (b).

To prove the statement about intersections in part (a), we use the fact that the kernel of the surjective map $I_1 \oplus I_2 \to (I_1 + I_2)$ that sends $i_1 \oplus i_2$ to $i_1 - i_2$ is precisely the image of the map $I_1 \cap I_2 \to I_1 \oplus I_2$ that sends i to $i \oplus i$. That is, we have a short exact sequence $0 \to I_1 \cap I_2 \to I_1 \oplus I_2 \to (I_1 + I_2) \to 0$. By part (b) here, $\mathrm{Tor}_j^T(T/(I_1 + I_2),\, R) = 0$ for $j \geq 1$, and by part (b) of the Exercises preceding the theorem, $\mathrm{Tor}_j^T(I_1 + I_),\, R) = 0$ for $j \geq 1$ as well. Thus, the sequence stays exact when we apply $\_ \otimes_t R$. But by part (a) of the preceding Exercises, each monomial ideal, when tensored with R, may be identified with its expansion to R. This implies that the sequence $0 \to (I_1 \cap I_2)R \to I_1R \oplus I_2R \to (I_1 + I_2)R \to 0$ is exact. But the kernel of the surjection $I_1R \oplus I_2R \to (I_1 + I_2)R$ is always $(I_1R) \cap (I_2R)$. It follows that $(I_1R) \cap (I_2R) = (I_1 \cap I_2)R$.

It remains to establish that the colon operation on monomial ideals commutes with expansion. We first consider the case where I_2 is principal, say $I_2 = \nu T$. Note that for any ideal I and element u, $\big(I : (u)\big)u = I \cap (u)$. Thus, $(I_1 :_T \nu T)\nu = I_1 \cap \nu T$, and we get $(I_1R :_R \nu R)\nu = I_1R \cap \nu R = (I_1 \cap \nu T)R = \big((I_1 :_T \nu T)\nu\big)R = (I_1 :_T \nu T)R\nu$. Since ν is not a zerodivisor in R, we get that $(I_1R :_R \nu R) = (I_1 :_T \nu T)R$, as required. In the general case, if $I_2 = (\nu_1, \ldots, \nu_k)T$, we have that $I_1R :_R I_2R = \bigcap_{j=1}^k (I_1R :_R \nu_jR) = \bigcap_{j=1}^k \big((I_1 :_T \nu_jT)R\big) = \big(\bigcap_{j=1}^k (I_1 :_T \nu_jT)\big)R = (I_1 :_T I_2)R$, as required. $\square$