715: Lectures of October 24, 2002

We now give the proof of the following result, stated last time:

Lemma. *Let R be a complete local domain of dimension d with perfect residue class field. Then the torsion free rank of ${}^eR \cong R^{1/q}$ as an R-module is q^d.*

Proof. Choose a coefficient field K for R and a system of parameters $x_1, \dots, x_d$, and then R is module-finite over $A = K[[x_1, \dots, x_d]]$. Let G and H be the fraction fields of A and R, respectively. The torsion-free rank of $R^{1/q}$ over R is the same as the degree of the field extension $[H^{1/q} : H]$ or of $[H : H^q]$. Now, $[H : G][G : G^q] = [H : H^q][H^q : G^q]$, since both are $[H : G^q]$, and the extension $G^q \subseteq H^q$ is isomorphic with $G \subseteq H$, so that $[H : G] = [H^q : G^q]$. Thus, we get that $[G : G^q] = [H : H^p]$, and it suffices to check the result for A instead of R. But $A = K[[x_1, \dots, x_d]]$ and $A^p = K[[x_1^q, \dots, x_d^q]]$, and A is free of rank q^d over A^q: K is perfect, $A^q \cong K[[x_1^q, \dots, mx_d^q]]$, and the monomials $x_1^{a_1} \cdots x_d^{a_d}$ with $0 \le a_i < q$ for all i form a free basis. $\square$

The following lemma is a special case of Monsky's theorem stated earlier: it is, in fact, the main case.

Lemma (Monsky). *Let R be a complete local domain of dimension d, with perfect residue field K, and let I be an m-primary ideal. Then $HK_I(e) = \gamma q^d + \mathrm{O}(q^{d-1})$ for a positive real constant γ.*

Proof. Let $a_e = HK_I(e)/q^d$. Since ${}^1R$ has torsion-free rank p^d as an R-module, we can choose an embedding $R^{\oplus p^d} \to {}^1R$ such that the kernel N is a torsion module over R, and so $\dim N \le d-1$. Then, for all e, we have an exact sequence

$$0 \to {}^eR^{\oplus p^d} \to {}^{e+1}R \to {}^eN \to 0.$$

Tensoring with R/I yields:

$$\cdots \to (R/I^{[q]})^{\oplus p^d} \to R/I^{[pq]} \to N/I^{[q]}N \to 0$$

which shows that $\ell(R/I^{[pq]} \le p^d(\ell(R/I^{[q]}) + \mathrm{O}(q^{d-1})$. Similarly, we can embed ${}^1R \subseteq R^{\oplus p^d}$ with torsion cokernel N' and find that $p^d\ell(R/I^{[q]}) \le \ell(R/I^{[pq]} + \mathrm{O}(q^{d-1})$. Thus, $|\,p^d\ell(R/I^{[q]}) - \ell(R/I^{[pq]})\,| = \mathrm{O}(q^{d-1})$. Dividing through by p^dq^d, we get that $|a_e - a_{e+1}| = \mathrm{O}(1/p^dq)$, and so is $\le C/q = C/p^e$ for some positive constant C. Thus, adding several of these inequalities for e, $e+1$, $\dots$, $e+k-1$ yields

$$|a_e - a_{e+k}| \le \frac{C}{p^e} + \cdots + \frac{C}{p^{e+k-1}} \le C(\frac{1}{p^e} + \frac{1}{p^{e+1}} + \cdots) = \frac{C}{1-\frac{1}{p}}(\frac{1}{p^e}) = \frac{C'}{p^e} = \frac{C'}{q}.$$

This proves that the a_e form a Cauchy sequence. Call the limit $\gamma \in \mathbb{R}$. Letting $k \to \infty$ we see that $|a_e - \gamma| \le C'/q$. Multiplying by q^d, we find that $|\mathrm{HK}_I(e) - \gamma q^d| \le C' q^{d-1}$, which gives the required result, except for the positivity of γ. But that follows at once from (4). $\square$

Example. Consider the ring $R = K[x,\, y,\, z]/(xy - z^m)$. It is spanned by monomials in x, y, and z such that the exponent on z is at most $m-1$. We want to compute $\mathrm{HK}_m(e)$, which is the length of $R/(x^q,\, y^q,\, z^q)$. We shall do this for all $q \ge m$. Note first that $R/(x^q,\, y^q)R$ is spanned by all monomials $x^i y^j z^k$ where $i,\, j < q$ and $k < m$. If $q = am + r$ where $0 \le r < m$, then z^q in the quotient is $x^a y^a z^r$. We must also kill the multiples of this element. The additional monomials to be killed are those with $i, j < q$ and $d < m$ such that $i, j \ge a$ and $k \ge r$ or such that $i, k \ge a+1$ (since we get $x^{a+1}y^{a+1}$ by multiplying $x^a y^a z^r$ by z^{m-r}). The monomials that survive are those such that $i < a$, $j < q$ and $k < m$, those such that $i < q$, $j < a$ and $k < m$, and those such that (i, j) is (a, a), $(a, a+1)$, or $(a+1, a)$ and $k < r$. The number surviving is $aqm + qam - aam + 3r = mq^2(2(a/q) - (a/q)^2 + 3r/q^2)$. Recall that $a = (q-r)/m$, where $r < m$. As $q \to \infty$, $a/q = (1 - r/q)/m \to 1/m$. It follows that $\gamma = 2 - 1/m$.

Theorem (first length test for tight closure). *Let $(R,\, m)$ be a local ring whose completion is reduced and equidimensional (it suffices, for example, if R is excellent, reduced and equidimensional), and such that R has a completely stable test element. (We shall show soon that this is true when R is excellent, local, and reduced.) Let $N \subseteq M$ be finitely generated R-modules, and suppose the $N \subseteq W \subseteq M$ is such that W/N has finite length. Then $W \subseteq N^*_M$ iff $W^{[q]}/N^{[q]}$ (where $W^{[q]}$, $N^{[q]} \subseteq F^e(M)$) is $\mathrm{O}(q^{d-1})$. (If R is local but does not necessarily satisfy any of the other hypotheses, it is still true that if $W \subseteq N{*}_M$, then $\ell(W^{[q]}/N^{[q]}) = \mathrm{O}(q^{d-1})$.)*

Proof. We first prove the final parenthetical statement: the converse is much harder. Suppose that $W \subseteq N^*_M$. Let $w_1, \, \ldots \,, w_h$ generate W. For each w_i we may choose $c_i \in R^\circ$ such that $c_i q_i^q \in N^{[q]}$ for all $a \gg 0$. Let c be the product of the c_i. Then $c \in R^\circ$, and $cW^{[q]} \subseteq N^{[q]}$ for all $q \gg 0$. Each w_i is killed by a power, say m^{t_i}, of m. Let t be the maximum of the t_i. Then W^q/N^q is the sum of the cyclic modules spanned by the images v_i of the w_i^q. Each v_i is killed by both $(m^t)^{[q]}$ and c. Since the number of v_i is h, we have that $\ell(W^{[q]}/N^{[q]}) \le h\ell(R/\bigl(Rc + (m^t)^{[q]}\bigr) = h\,\mathrm{HK}_{m^t}(R/Rc) = h\mathrm{O}(q^{d-1})$, as required, since $\dim{(R/Rc)} \le d-1$.

For the converse, it suffices to show that if $w \in W$ but $w \notin N^*$ then the length of $(N^{[q]} + Rw^q)/N^{[q]}$ is $\ge cq^d$ for some constant $c > 0$. We can pass to the completions of R, M, and N without affecting any relevant issues, and represent R as a module-finite extension of A, a complete regular local ring with the same residue field. The extension may not be separable, but we can choose q'' so large that $S = R[A^{1/q''}] \subseteq R^{1/q''}$ is separable over $A^{1/q''}$. (This means that when one tensors with the fraction field of $A^{1/q''}$, the result is a product of finitely many separable algebraic extensions of the fraction field of $A^{1/q''}$.) By a slight generalization of what we did earlier when constructing test elements, one can choose $d \in A^\circ$ such that $dS^{1/q} \subseteq S[A^{1/qq''}]$ for all $q > 0$. Also, $S[A^{1/qq''}] \cong S \otimes_{A^{1/q''}} A^{1/qq''}$ is flat over S.

We may assume without loss of generality that M is free. (Replace it by a free module that maps onto it, replace N by its inverse image in that free module, and replace w be an element of the free module that maps onto it.)

We can choose q' such that $dw^q \notin (N^{[q]})^*_M$ for all $q \geq q'$ by an earlier lemma (for if $dw^q \in (N^{[q]})^*_N$ for infinitely many q, then $w \in N^*$, by an earlier lemma). Then $dw^q \notin N^{[q]}R^\infty$ all $q \geq q'$, since $N^{[q]}R^\infty \cap M \subseteq (N^{[q]})^*_M$ (in fact, $N^{[q]}R^\infty \cap M$ is the same as the Frobenius closure of $N^{[q]}$ in M when R is reduced).

For all powers Q of p, let ${}_QJ$ denote the ideal $\{a \in A : au^Q \in N^{[Q]}\}$. Then $A/{}_QJ$ injects, as an A-submodule, into $W^{[Q]}/N^{[Q]}$: simply send the class of a to the image of aw^Q. Since A and R have the same residue class field, length over R agrees with length over A, and it suffices to show that $\ell(A/{}_QJ) \geq cQ^d$ for some sufficiently small positive real number c and all $Q \gg 0$.

Assume $Q \geq q'q''$, and write $Q = qq'q''$. Then $a \in {}_QJ$ implies that $au^Q \in N^{[Q]}$, and so $a^{1/qq''}w^{q'} \in N^{[q']}R^{1/qq''}$. Multiply by d: we then have $da^{1/qq''}w^{q'} \in N^{[q']}S[A^{1/qq''}] \cong N^{[q']}S \otimes_S S[A^{1/qq''}]$. Therefore,

$$a^{1/qq'} \in (N^{[q']}S \otimes_S S[A^{1/qq''}]) :_{S[A^{1/qq''}]} dw^{q'} \cong (N^{[q']}S :_S dw^{q'})S[A^{1/qq''}],$$

by the flatness of $S[A^{1/qq''}]$ over S. Now, $N^{[q']}S :_S dw^{q'} \subseteq N^{[q']}R^{1/q''} :_{R^{1/q''}} dw^{q'}$, since $S \subseteq R^{1/q''}$. But, $dw^{q'} \notin N^{[q']}R^{1/qq''}$, and so $N^{[q']}S :_S dw^{q'} \subseteq m^{1/q''}$, the maximal ideal of $R^{1/q''}$. Then $a^{1/(qq'')} \in m^{1/(qq'')}$, and so, taking qq'' powers, $a \in m^{[q]}$. Thus, ${}_QJ \subseteq m^{[Q/(q'q'')]}R \cap A \subseteq m^{Q/(q'q'')}R \cap A$ for all $Q \geq q'q''$.

Let m_A denote the maximal ideal of A. We can choose B to be a power of p such that such that $m^B \subseteq m_A R$, since $m_A R$ is m-primary. It now follows that ${}_QJ \subseteq m_A^{Q/(q'q''B)} \cap A$ for all $Q \geq q'q''B$. R is a finitely generated A-module and A is a submodule of R. Thus, by the Artin-Rees lemma, there exists a constant f such that $m_A^n R \cap A \subseteq m_A^{n-f}$ for all $n \gg 0$. Thus, with $D = q'q''B$, we have that ${}_QJ \subseteq m_A^{(Q/D)-f}$ for all $Q \gg 0$, and so $\ell(A/{}_QJ \geq \ell(A/m_A^{(Q/D)-f})$. This last is a value of the usual Hilbert function, and so, for $Q \gg 0$, is a polynomial of degree d with positive leading coefficient in $(Q/D) - f$. But this is still a polynomial of degree d in Q with positive leading coefficient, asymptotic to cQ^d for some positive constant c, and the result follows. $\square$

Corollary (second length test for tight closure). *Let R be as in the preceding theorem and let $I \subseteq J$ be m-primary ideals. Then $J \subseteq I^*$ if and only if the Hilbert-Kunz multiplicities γ_I and γ_J are equal.*

Proof. This is straightforward, from Monsky's theorem and the preceding theorem, for $\ell(J^{[q]}/I^{[q]}) = \ell(R/I^{[q]}) - \ell(R/J^{[q]}) = \gamma_I q^d - \gamma_J q^d + \mathrm{O}(q^{d-1}) - \mathrm{O}(q^{d-1}) = (\gamma_I - \gamma_J)q^d + \mathrm{O}(q^{d-1})$, where, clearly, $\gamma_I \geq \gamma_J$. Evidently, this is $\mathrm{O}(q^{d-1})$ if and only if $\gamma_I - \gamma_J = 0$. $\square$