715: Lecture of October 28, 2002

We conclude our discussion of Hilbert-Kunz functions, at least for the moment, by mentioning one more example, due to Han and Monsky. Let $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$, and let $R = \mathbb{Z}_5[x_1, x_2, x_3, x_4]/(x_1^4+x_2^4+x_3^4+x_4^4)$. Then $\ell(R/m^{[5^e]}) = \dfrac{168}{61}(5^e)^3 - \dfrac{107}{61}(3^e)$. The details are difficult and we will not give them here.

Our next objective is to investigate the *persistence* of tight closure. We want it to be true that for any homomorphism of Noetherian rings $f : R \to S$, if $u \in I^*$, then $f(u) \in (IS)^*$.

More generally, we want it to be true that if $N \subseteq M$ are finitely generated modules such that $u \in N^*_M$, then the image $1 \otimes u$ of u in $S \otimes_R M$ is in the tight closure, over S, of the image of $S \otimes_R N$ in $S \otimes_R M$. When this is the case, we shall say that *persistence* (of tight closure) holds for the ring homomorphism $R \to S$ The problem is that the element $c \in R^\circ$ used in the tight closure test that places u in I^* or N^*_M may not map into S°: it may even map to 0 in S.

We make the following elementary observations about this problem.

(1) In considering whether persistence holds for $R \to S$, it suffices to consider the issue when M is free and $N \subseteq M$.

(2) If R° maps into S° then persistence holds. For example, if $R \subseteq S$ are domains then persistence holds. If $R \to S$ is flat, persistence holds, and, in particular, it holds if S is a localization of R. (A minimal prime Q of S lies over a minimal prime P of R in the flat case, since $R_P \to S_Q$ is faithfully flat, and so the nilpotence of QS_Q implies the nilpotence of PR_P.)

(3) If S is the quotient of R by a minimal prime, then persistence holds. (Any element of R° maps to a nonzero element in the domain S.)

(4) If persistence holds for finitely many ring homomorphisms then it holds for their composition.

(5) If R has a test element that is not in the prime ideal Q, then persistence holds for $R \to R/Q$.

The following is very useful:

Lemma. *If S is a domain which is a module-finite extension of the domain R, then $u \in N^*_M$ if and only if $1 \otimes u$ is in the tight closure of the image of $S \otimes_R N$ in $S \otimes_R M$.*

Proof. The "only if" part is clear from the first statement of (2) above. Now suppose that u in M is such that $1 \otimes u$ is in the tight closure of the image of N in M. We assume, as we may that, that M is free, and think of $S \otimes M$ as MS, and the image of $S \otimes N$ as NS, the submodule generated by the image of N. Forming NS is analogous to expanding an ideal. Let d be the nonzero element used in the tight closure test which shows that $u \in M$ (since $M \subseteq MS$, we shall simply write u instead of $1 \otimes u$). Then d has a nonzero multiple c in R, and then we have that $cu^q \in (NS)^{[q]} = N^{[q]}S$ for all $q \gg 0$. We can choose an R-linear map $S \to R$ such that $1 \in S$ maps to $c_1 \neq 0$ in R. (Do this first after tensoring with the fraction field of R and then clear denominators.) Applying this map coordinate-wise shows that $c_1 cu^q \in N^{[q]}$ for all $q \gg 0$. □

The following theorem will be extremely useful in establishing persistence. It will apply right away to show that persistence holds for maps from an F-finite ring, and we shall later use it to prove the same thing for algebras essentially of finite type over an excellent local ring.

Theorem (persistence in the presence of many test elements). *Let $\mathcal{C}$ be a class of Noetherian rings of positive prime characteristic such that*

(i) *The class is closed under taking quotients by prime ideals.*
(ii) *If a domain is in $\mathcal{C}$, then so is its integral closure in its fraction field.*
(iii) *If R is a domain in $\mathcal{C}$, and $c \in R$ is a nonzero element such that R_c is regular, then c has a power that is a test element.*
(iv) *The non-regular locus in each domain in $\mathcal{C}$ is Zariski closed.*

Then persistence of tight closure holds for $R \to S$ for any ring R in $\mathcal{C}$ and Noetherian ring S.

Proof. If we have an instance of the failure of persistence, then we have such an instance with S a domain, since tight closure will also fail modulo some minimal prime of S. Thus, we may assume that S is a domain. Now some minimal prime P of R will be contained in $\operatorname{Ker} R \to S$, and so we may first make the transition from R to R/P, by observation (3) on persistence stated earlier. Thus, we may assume that persistence fails for a map from a domain in $\mathcal{C}$ to a domain. Let S_0 be the image of R in S. The persistence must fail when we pass from R to S_0, since it holds for $S_0 \subseteq S$. Thus, persistence must fail in the mapping from $R \to R/Q$ for some domain R in $\mathcal{C}$ and for some prime Q in R. Consider the successive quotients of R by the primes in a saturated chain from (0) to Q. Then persistence must fail for some link in the chain. Therefore, there must be a domain R in $\mathcal{C}$ and a height one prime Q in R such that persistence fails for the map $R \to R/Q$.

Let S be the integral closure of R in its fraction field. Then some prime ideal Q' of S lies over Q, and S/Q' is evidently a module-finite extension domain of R/Q. Moreover, Q' must have height one. (A chain of distinct primes in the larger ring of an integral extension contracts to a chain of distinct primes in the smaller ring, because the primes lying over a given prime are mutually incomparable.)

Tight closure persists from S to S/Q' because Q' is a height one prime of S and cannot contain all the test elements in S: because S is normal, either it is regular, or there is a regular sequence of length 2, call it c, d, such that S_c and S_d are both regular. If S is regular, then $1 \notin Q'$ is a test element. Otherwise, at least one of c, d is not in Q', say c, and then c has a power that is a test element. Since $c \notin Q'$, observation (5) shows that persistence holds for $S \to S/Q'$. Therefore, persistence holds for the composition $R \to S/Q'$. Since S/Q' is module-finite over R/Q, the fact that an element is in a tight closure after expansion to S/Q' implies that it is in the the tight closure over R/Q, by the lemma preceding the theorem, and it follows that persistence holds for $R \to R/Q$, as required. $\square$

Now F-finite rings are excellent, so that the integral closure of an F-finite domain is a finite module over it, and therefore again F-finite. It follows that F-finite rings satisfy the hypotheses on the class $\mathcal{C}$ above, and so:

Corollary. *If R is F-finite, then persistence holds for $R \to S$ for any homomorphism from R to a Noetherian ring S.* $\square$

We next want to prove that persistence holds for rings essentially of finite type over an excellent local ring R. To this end, we want to prove the following result, which will require a fair amount of work.

Theorem. *Let R be a reduced ring essentially of finite type over an excellent local ring B, and let $c \in R^\circ$ be such that R_c is regular. Then c has a power that is a test element.*

Note that we may replace B by its image in R, and therefore that we may assume that B is reduced.