715: Lecture of October 31, 2002

Here is one more remark about Hilbert Kunz functions that I meant to make earlier. Let (R, m) be a regular local ring of Krull dimension d, and let I be any m-primary ideal. Let $\ell(R/I) = h$. Then $\mathrm{HK}_I(e) = hq^d$ for all $e \geq 0$. The point is that because R is regular, the iterated Frobenius functors are flat. Thus, if R/I has a filtration in which the factors are h copies of $K = R/m$, then $R/I^{[q]} \cong F^e(R/I)$ has a filtration in which the factors are h copies of $F^e(R/m) = R/m^{[q]} = R/(x_1^q, \ldots, x_d^q)$, where $x_1, \ldots, x_d$ is a system of parameters that generates m, and $\ell\big(R/(x_1^q, \ldots, x_d^q)\big) = q^d$.

Recall that last time we asserted that our next goal is to prove:

Theorem. *Let R be a reduced ring essentially of finite type over an excellent local ring B, and let $c \in R^\circ$ be such that R_c is regular. Then c has a power that is a test element.*

Note that we may replace B by its image in R, and therefore that we may assume that B is reduced. In fact, we shall ultimately prove this under the weaker hypothesis that R_c is weakly F-regular and Gorenstein. However, we shall first focus on the case where R_c is regular.

In order to prove this, we first prove the following:

Theorem. *Let R be a reduced ring essentially of finite type over an excellent local ring $B \subseteq R$, and let $c \in R^\circ$ be such that R_c is regular. Then B has a faithfully flat local extension ring C such that C is F-finite and $(C \otimes R)_c$ is weakly F-regular and Gorenstein.*

The proof of this theorem requires some substantial effort. The first step is to replace B by its completion $\widehat{B}$, and this is no problem so long as we are working with the condition of regularity.

We first recall the notion of a p-base for a field K of characteristic $p > 0$. Call a set S of elements *p-independent* if for each finite subset $s_1, \ldots, s_h$ of S with h distinct elements, we have that $[K^p[s_1, \ldots, s_h] : K^p] = p^h$. Since the pth powers of the elements adjoined are in K^p, this is the maximum possible degree. By Zorn's lemma, there is a maximal such set of elements, and we call such a set a p-base. If Λ is a p-base, then $K = K^p[\Lambda]$. We also have that $K^{1/p} = K[\lambda^{1/p} : \lambda \in \Lambda]$. It is also true that for any h distinct elements $\lambda_1, \ldots, \lambda_h$ of the p-base and for all q, $[K^q[\lambda_1, \ldots, \lambda_h] : K^q] = q^h$ and that $K^{1/q} = K[\lambda^{1/q} : \lambda \in \Lambda]$. If K is perfect, the empty set is the only p-base.

Now suppose that R is a complete local ring with coefficient field $K \subseteq R$. Let Λ be a p-base for K. Let Γ be a subset of Λ that contains all but finitely many elements of Λ. We shall say the Γ is a *cofinite* subset of Λ. We want to describe a construction that gives a faithfully flat purely inseparable extension R^Γ of R whose residue field, K^Γ, contains qth roots of all elements of Γ for all q. It will also turn out that R^Γ is an F-finite local

Noetherian ring. Moreover, in many instances, a specific "good" property of R (or of a specific R-algebra) can be preserved when we pass to R^Γ (or apply $R^\Gamma \otimes_R \_$) *provided* that we choose Γ to be a sufficiently small cofinite subset of Λ. We shall not give all details of all proofs, but we do want to give a very clear idea of what is involved.

We begin by describing what happens in the case of a complete regular local ring $A = K[[x_1, \dots, x_d]]$ of Krull dimension d. For each $q = p^e$ let K_e^Γ denote the field $K[\gamma^{1/p^e} : \gamma \in \Gamma]$. Let A_e^Γ denote the ring $K_e^\Gamma[[x_1, \dots, x_d]]$. This ring is purely inseparable over A, and faithfully flat over it.

Now $K\Gamma_e \subseteq K\Gamma_{e'}$ if $e \le e'$, and so $A\Gamma_e \subseteq A\Gamma_{e'}$, and we may form the non-decreasing union $A^\Gamma = \bigcup_e A_e^\Gamma$. The successive inclusions are all faithfully flat and purely inseparable. The union is therefore a faithfully flat and purely inseparable extension of A. It is no longer complete, however. A key point is that this ring is Noetherian. This follows from the following:

Lemma. *Suppose that $\{K_e\}_e$ be any non-decreasing sequence of fields, and let $A_e = K_e[[x_1, \dots, x_d]]$. Let $L = \bigcup_e K_e$. Then $A = \bigcup_e A_e$ is a regular Noetherian ring of Krull dimension d with residue field L, faithfully flat over each A_e, such that $x_1, \dots, x_d$ is a system of parameters generating the maximal ideal. Moreover, $\widehat{A} \cong L[[x_1, \dots, x_d]]$.*

Proof. The only hard step is to prove that A is Noetherian. Let $B = L[[x_1, \dots, x_d]]$, which is faithfully flat over every A_e. Let I_t denote a non-decreasing chain of finitely generated ideals in A. It suffices to show that it is eventually stable. Since B is Noetherian, the chain I_tB is eventually stable. Suppose $I_tB = I_{t+1}B = \cdots$. It suffices to prove that $I_t = I_{t+1} = \cdots$. Suppose, for example, that $IB = JB$ but $I \neq J$. Choose e so large that A_e contains generators for both I and J in A. Then we can choose I_0, J_0 contained in A_e such that $I_0A = I$ and $J_0A = J$. But then $I_0B = IB = JB = J_0B$. Since B is faithfully flat over A_e, it follows that $I_0 = J_0$, and so $I = J$. It is clear that $x_1, \dots, x_d$ generate the maximal ideal of A, which is the union of the maximal ideals of the A_e, and that they form a regular sequence. Thus, they are a regular system of parameters for A. The remaining statements now follow easily. $\square$

A critical point is:

Theorem. A^Γ *is F-finite.*

Proof. Let $\theta_1, \dots, \theta_h$ be the finitely many elements that are in the p-base Λ but not in Γ. We shall show that the monomials in $\theta_1^{1/p}, \dots, \theta_h^{1/p}$ and $x_1^{1/p}, \dots, x_d^{1/p}$ such that the degree on each element is at most $< p$ span $(A^\Gamma)^{1/p}$ as an A^Γ-module. First note that $(A^\Gamma)^{1/p} = \bigcup_e (A_e^\Gamma)^{1/p}$, and for every e, $(A_e^\Gamma)^{1/p} = K_e^{1/p}[[x_1^{1/p}, \dots, x_d^{1/p}]]$. This is spanned over $(K_e^\Gamma)^{1/p}[[x_1, \dots, x_d]]$ by the specified monomials in the $x_j^{1/p}$, and so the result will follow provided that the set T of monomials in $\theta_1^{1/p}, \dots, \theta_h^{1/p}$ of degree$< p$ in $(K_e^\Gamma)^{1/p}$ over K_{e+1}^Γ for every e, for that shows that for all e, they span $(A_e^\Gamma)^{1/p}$ over A_{e+1}^Γ. But $(K_e^\Gamma)^{1/p}$ is spanned over K by products of monomials in T and monomials in the elements $\gamma^{1/qp}$ for $\gamma \in \Gamma$, and the result now follows from the fact that the latter are in K_{e+1}^Γ. $\square$

We now want to extend this construction to complete local rings R that need not be regular. We assume that a coefficient field $K \subseteq R$ and a cofinite subset Γ of a p-base for K have been specified. One method of extending the construction is to choose $A = K[[x_1, \ldots, x_d]] \subseteq R$ (where the copy of K used is the given coefficient field), and then let $R^\Gamma = A^\Gamma \otimes_A R$. This works, but we want to see that it is actually independent of the choice of A, and so it is desirable to have a different description.