715: Lecture of October 31, 2002

We can give a more invariant description of the Γ construction as follows. Let R be a complete local ring of characteristic $p > 0$ with coefficient field K, and let Γ be cofinite in a p-base for K. Then we may define R_e^Γ as follows: form the ordinary tensor product $K_e^\Gamma \otimes_K R$, and then complete with respect to the maximal ideal $\mathcal{M}_R$ that is the kernel of the map $K_e^\Gamma \otimes_K R \to K_e^\Gamma$ that sends $\alpha \otimes r$ to $\alpha\rho$, where ρ is the residue class of r in $K \subseteq K_e^\Gamma$. One can verify that if $R = K[[x_1, \dots, x_d]]$ then this gives the same notion as before. One then verifies as well that when $A = K[[x_1, \dots, x_d]] \subseteq R$ have the same coefficient field K, and $\Gamma \subseteq K$ is cofinite in a p-base, then

$$\widehat{(K_e^\Gamma \otimes_K A)}^{\mathcal{M}_A} \otimes_A R \cong \widehat{(K_e^\Gamma \otimes_R)}^{\mathcal{M}_R}.$$

There are two points that should be made here: on the right, it does not matter whether we complete with respect to $\mathcal{M}_R$ or with respect to the expansion of $\mathcal{M}_A$, since the maximal ideal of R is nilpotent modulo the expansion of the maximal ideal of A. Also, because R is module-finite over A, $\widehat{(K_e^\Gamma \otimes_K A)}^{\mathcal{M}_A}$ on the left remains complete when we apply $\_ \otimes_A R$. Of course, once we have a construction for R_e^Γ, we can define R^Γ by taking a union over all e.

We next digress to prove some basic facts about the behavior of depth and type when one has a flat morphism. We first want to recall the following:

Fact. *If $x_1, \dots, x_k$ is a possibly improper regular sequence (resp., a regular sequence) on R and S is flat (resp., faithfully flat) over R, then $x_1, \dots, x_k$ is also a possibly improper regular sequence (resp., a regular sequence) on S.*

Proof. If $0 \to R \xrightarrow{x} R$ is exact it remains exact when we apply $S \otimes_R \_$. This handles the case when $k = 1$, and the general case follows by induction: note that if S is flat over R and $x \in R$, then S/xR is flat over R/xR. If $I = (x_1, \dots, x_k)R \neq R$, then $R/I \neq 0$, and so, if S is faithfully flat, $S \otimes_R R/I \cong S/IS \neq 0$. $\square$

We next note the following:

Lemma. *Let $(R, m) \to (S, n)$ be a flat local homomorphism of local rings. Let $x \in S$ be an element that is a nonzerodivisor on the closed fiber, S/mS. Then x is a nonzerodivisor on S/IS for every ideal I of S, and S/xS is again flat over R.*

Proof. To prove the first statement, assume that it is false, and choose I maximal so that it gives a counterexample. We may replace R by R/I, S by $S/IS = (R/I) \otimes_R S$, which is flat over R/I, and x by its image in S/IS. The closed fiber does not change. Thus, we simply need to prove that if x is a nonzerodivisor on S/mS, then it is a nonzerodivisor on S, and we may assume $(*)$ that x is a nonzerodivisor on S/JS for every nonzero ideal J of R. We consider two cases. First, suppose that there is a nonzerodivisor y in m. Then, since S is flat over R, y is a nonzerodivisor on S. By the assumption $(*)$, x is a nonzerodivisor on S/yS, so that $y,\, x$ is a regular sequence in S. But, in a local ring, regular sequences are permutable. Thus, x is a nonzerodivisor on S, as required. The second case is that m consists entirely of zerodivisors, which implies that m is an associated prime of R. Thus, there is an element $u \in R$ whose annihilator is m. If $u \notin m$, then R must be a field, and S is identical with the closed fiber. Thus, we may assume that $u \in m$. By the assumption $(*)$, x is a nonzerodivisor on S/uS, and it is also a nonzerodivisor on $uS \cong S \otimes_R uR \cong S \otimes_R K = S/mS$. But if x is a nonzerodivisor on both N and M/N, it is a nonzerodivisor on M. Thus x is a nonzerodivisor on S in this second case as well.

Next, to show that S/xS is R-flat it suffices to show that $\mathrm{Tor}_1^R(M,\, S/xS) = 0$ for every R-module M. By a direct limit argument, it is enought to consider the case of finitely generated modules, and since each such module has a finite filtration in which the factors are cyclic modules, it suffices to show that $\mathrm{Tor}_1^R(R/I,\, S/xS) = 0$ for every ideal I of R. Consider the short exact sequence $0 \to S \xrightarrow{x} S \to S/xS \to 0$ and apply $R/I \otimes_R \_\_$. Since S is R-flat, $\mathrm{Tor}_2^R(R/I,\, S) = 0$, and this yields an exact sequence $0 \to \mathrm{Tor}_1^R(R/I,\, S/xS) \to S/IS \xrightarrow{x} S/IS$. Since x is not a zerodivisor on S/IS, $\mathrm{Tor}_1^R(R/I,\, S/xS) = 0$, as required. $\square$

Corollary. *Let $(R, m) \to (S, n)$ be a flat local homomorphism of local rings. Suppose that $x_1,\, \ldots\, , x_k \in n$ is a regular sequence on the closed fiber, S/mS. Then $x_1,\, \ldots\, , x_k$ is a regular sequence on S/IS for every ideal I of S, and $S/(x_1,\, \ldots\, , x_k)S$ is again flat over R.*

Proof. The result is immediate by induction on k from the preceding lemma. $\square$

The following result will prove very useful:

Theorem. *Let $(R, m) \to (S, n)$ be a flat local homomorphism of local rings.*
(a) *Minimal primes of S lie over minimal primes of R (even when the rings are not necessarily local).*
(b) *$\dim S = \dim R + \dim S/mS$.*
(c) *$\operatorname{depth} S = \operatorname{depth} R + \operatorname{depth} S/mS$, where the depth of a local ring is the length of any maximal regular sequence in the maximal ideal.*
(d) *S is Cohen-Macaulay if and only if both R and S/mS are Cohen-Macaulay.*
(e) *If S is Cohen-Macaulay (so that both R and S/mS are), then the type of S is the product of the types of R and S/mS.*
(f) *S is Gorenstein if and only if both R and S/mS are Gorenstein.*

Proof. (a) If Q is minimal in S and lies over P in R, then $R_P \to S_Q$ is faithfully flat. Since QS_Q is nilpotent, so is PR_P.

For part (b) use induction on $\dim R$. If $\dim R = 0$ then the maximal ideal of R is nilpotent. Killing its expansion to S does not change the dimension of either ring. R is replaced by its residue field, S by the closed fiber (which is still the closed fiber), and the result is clear. If $\dim R > 0$, choose an element of R not in any minimal prime. Then from

(a) this element is not in any minimal prime of S. Now consider $R/xR \to S/xS$, which is still flat. The dimensions of both rings have decreased by one, while the closed fiber is unaffected. The result now follows from the induction hypothesis.

For part (c) we may use the Fact stated earlier to kill a maximal R-sequence in m, which will also be a regular sequence in S. The depths of both rings decrease by the same amount, which is $\operatorname{depth} R$, while the closed fiber is unaffected. Thus, we may assume without loss of generality that $\operatorname{depth} R = 0$. Now lift a maximal regular sequence in the maximal ideal of the closed fiber to a sequence of elements in n. By the Corollary to the Lemma proved earlier, this is a regular sequence in S, and if we kill it the depths of both S and S/mS decrease by the same amount, equal to $\operatorname{depth} S/mS$, while flatness is preserved. We have now reduced to showing that when R and S/mS both have depth 0, so does S. Since R has depth 0, we may embed $0 \to R/m \to R$. Applying $\_ \otimes_R S$, we get an embedding $0 \to S/mS \to S$. Since S/mS has depth 0, we have an embedding $0 \to S/n \to S/mS$, and the composition gives an embedding $0 \to S/n \to S$, which shows that S has depth 0.

Now (d) follows from the fact that the depth of any local ring is at most its dimension, that equality holds if and only if the ring is Cohen-Macaulay, together with parts (b) and (c) above.

To prove (e), note that we can kill a maximal regular sequence in R without affecting the issue, and that we can kill the lift of a maximal regular sequence in S/mS to n without affecting the issue. Thus we may assume that R and S/mS are Artinian (and S will be Artinian as well). let $V \cong (R/m)^{\oplus a}$, where a is the type of R, denote the socle in R, which is the same as $\operatorname{Ann}_R m = (0) :_R m$. The calculation of colons of ideals was shown earlier to commute with flat base change when the denominator ideal is finitely generated. Thus, $\operatorname{Ann}_S mS = (0) :_S mS = \big((0) :_R m\big)S$ which we may identify with $\big((0) :_R m\big) \otimes_R S$, since S is R-flat. This is $V \otimes S \cong (R/m)^{\oplus a} \otimes S \cong (S/mS)^{\oplus a}$. Now, the socle in S is clearly contained in $\operatorname{Ann}_S m$ which we just showed may be identified with $(S/mS)^{\oplus a}$. Therefore the socle in S may be identified with the annihilator of n in $(S/mS)^{\oplus a}$, which is obviously the direct sum of a copies of $\operatorname{Ann}_{S/mS} n \cong (S/n)^{\oplus b}$, where b is the type of S/mS. This shows that the socle in S is $\cong \big((S/n)^{\oplus b}\big)^{\oplus a} \cong (S/n)^{\oplus ab}$, as required. Part (f) follows at once from this, the fact the Gorenstein local rings are precisely the Cohen-Macaulay local rings of type 1, and the fact that the product of two positive integers is 1 if and only if both of them are equal to 1. $\square$

Since the issues are local on S:

Corollary. *Let $R \to S$ be flat. If R and all the fibers are Cohen-Macaulay, then S is Cohen-Macaulay. If R and all the fibers are Gorenstein, then S is Gorenstein.* $\square$