715: Lectures of November 4 and 6, 2002

Proposition. *Let $(R, m) \to (S, n)$ be a flat local homomorphism of Noetherian rings of prime characteristic $p > 0$ such that $n = mS$, R is weakly F-regular and Gorenstein, and S is purely inseparable over R. Then S is weakly F-regular and Gorenstein.*

Proof. Since R is Gorenstein and the fiber is $S/mS = S/n$, a field, S is Gorenstein of the same dimension as R. Thus, a system of parameters $x_1, \ldots, x_d$ for R will also be a system of parameters in S. It suffices to show that this system of parameters generates a tightly closed ideal J in S. Let $u \in R$ generate the socle modulo $I = (x_1, \ldots, x_d)R$, so that $I :_R m = I + Ru$. Then since S is R-flat, we have that $IS_S : mS = IS + Su$, i.e., that $J :_S n = IS + Su$, and since, in fact, S is faithfully flat over R, $u \notin IS = J$. It follows that u generates the socle in S/J, and so if J is not tightly closed, we must have $u \in J^*$. Then for some $c \in S^\circ$, we have that $cu^q \in \big((x_1, \ldots, x_d)S\big)^{[q]}$ for all $q \gg 1$. By replacing c by a q_0 power for some large q_0 we may assume that $c \in R$, and then we have that for all $q \gg 1$, $cu^q \in (x_1^q, \ldots, x_d^q)S \cap R = I^{[q]}S \cap R = I^{[q]}$, since S is faithfully flat over R. Thus, $cu^q \in I^{[q]}$ for all $q \gg 1$, a contradiction, because we know that $I = I^*$ in R and $u \notin I$. □

Lemma. *Let K be a field of characteristic $p > 0$ and let Λ be a p-base for K. The family of subfields K^Γ as Γ runs through the cofinite subsets of Λ is directed by $\supseteq$, and the intersection of these fields is K.*

Proof. K^∞ has as a basis 1 and all monomials $\lambda_1^{\alpha_1} \cdots \lambda_t^{\alpha_t}$ where t is some positive integer, $\lambda_1, \ldots, \lambda_t$ are mutually distinct elements of Λ, and the α_j are positive rational numbers in $(0, 1)$ whose denominators are powers of p. If u were in the intersection and not in K it would have a unique representation as a K-linear combination of these elements, including at least one monomial μ as above other than 1. Choose $\lambda \in \Lambda$ that occurs in this monomial with positive exponent. Then that monomial is not in K^Γ, which has a basis consisting of 1 and all such monomials such that the λ_j occurring are in Γ. It follows that $u \notin K^\Gamma$. □

Theorem. *Let $\{L_j\}_j$ be any family of fields directed by $\supseteq$ with $\bigcap_j L_j = K$. Let $\mathcal{F}_j$ be the fraction field of $A_j = L_j[[x_1, \ldots, x_d]]$, and let $\mathcal{F}$ be the fraction field of $A = K[[x_1, \ldots, x_d]]$ (also written $K((x_1, \ldots, x_d))$). Then $\bigcap_j \mathcal{F}_j = F$. Consequently, $\bigcap_\Gamma K^\Gamma((x_1, \ldots, x_d)) = \bigcap_\Gamma \mathrm{frac}\, A^\Gamma = \mathcal{F}$ as Γ runs through cofinite subsets of a p-base for K.*

Similarly, $\bigcap_j L_j(x_1, \ldots, x_d) = K(x_1, \ldots, x_d)$.

Proof. The final statement is completely elementary. By induction on d, it reduces to the case where $d = 1$. But any element of any $L_j(x_1, \ldots, x_d)$ can be written uniquely in lowest terms with a monic polynomial in the denominator, and this representation is independent of which L_j one is thinking about. Hence, when represented this way, an element of the intersection must have all coefficients in all of the L_j.

To prove the first statement, we introduce some terminology. An algebra C over B is called *injectively free* over B if for every $u \neq 0$ in C there is an element $h \in \mathrm{Hom}_B(C, B)$ such that $h(u) \neq 0$. In other words, the natural map of C into its double dual over B is injective. Note that if C is injectively free over B then $C[x_1, \ldots, x_d]$ is injectively free over $B[x_1, \ldots, x_d]$, and that $C[[x_1, \ldots, x_d]]$ in injectively free over $B[[x_1, \ldots, x_d]]$: choose a nonzero coefficient of u, choose a map $C \to B$ which is nonzero on that coefficient, and then extend it by letting it act on coefficients.

We next observe the following fact: $(*)$ if $B \subseteq C$ are domains, with fraction fields $\mathcal{K} = \operatorname{frac} B$ and $\mathcal{L} = \operatorname{frac} C$, respectively, C is injectively free over B, and x is a formal indeterminate over C, then $(\operatorname{frac} C[[x]]) \cap \mathcal{K}((x)) = \operatorname{frac} B[[x]]$. It suffices to show $\subseteq$: the other inclusion is obvious. Suppose that $u \in \operatorname{frac} C[[x]] \cap \mathcal{K}((x)) - \{0\}$. Then we can write it as $x^h(\sum_{j=0}^{\infty} s_j x^j)$ where $h \in \mathbb{Z}$ with the $s_j \in \mathcal{K}$. All three fields contain the powers of x, and so we may multiply by x^{-h} without affecting the issues. Thus, we may assume that $h = 0$. We want to show that $u \in \operatorname{frac} B[[x]]$. Since $u \in \operatorname{frac} C[[x]]$, there exists $v \neq 0$ in $C[[x]]$ such that $w = vu \in C[[x]]$, say $v = \sum_{j=0}^{\infty} c_j x^j$ and $w = \sum_{k=0}^{\infty} d_k x^k$, where the $c_j,\, d_j \in C$. Then for each $m \geq 0$, we have that $\sum_{j+k=m} c_j s_k = d_m$. Choose j_0 such that $c_{j_0} \neq 0$ and choose $f : C \to B$, B-linear, such that $f(c_{j_0}) \neq 0$ in B. Extend f to a map $C[[x]$ to $B[[x]]$ by letting it act on coefficients. Then $\sum_{j+k=m} f(c_j) s_k = f(d_m)$, for every m (the s_k are in $\mathcal{K}$ rather than in B, but one may multiply by a single element of $B - \{0\}$ to clear denominators, get the desired result after multiplication, and then divide). This shows that $f(v)u = f(w)$, and $f(v) \neq 0$ because $f(c_{j_0}) \neq 0$. Since $f(v),\, f(w) \in B[[x]]$, we have that $u = f(w)/f(v) \in \operatorname{frac} B[[x]]$, as required.

We can now prove the theorem by induction on d. If $d = 1$ it follows from the uniqueness of coefficients in the Laurent expansion of an element of $L_j((x)) = L_j[[x]][x^{-1}]$. For the inductive step, let $B = K[[x_1, \ldots, x_{d-1}]]$, $C_j = K_j[[x_1, \ldots, x_{d-1}]]$, $\mathcal{K} = \operatorname{frac} B$ as above and $\mathcal{L}_j = \operatorname{frac} C_j = K_j((x_1, \ldots, x_{d-1}))$. Then $\bigcap_j K_j((x_1, \ldots, x_d)) \subseteq \bigcap_j \mathcal{L}_j((x_d)) = (\bigcap_j \mathcal{L}_j)((x_d))$ (by the case where $d = 1$) $= \mathcal{K}((x_d))$ (by the induction hypothesis). For any fixed j_0, $\bigcap_j K_j((x_1, \ldots, x_d)) \subseteq \operatorname{frac}(C_{j_0}[[x_d]]) \cap \mathcal{K}((x_d)) = \operatorname{frac} B[[x_d]] = K((x_1, \ldots, x_d))$, as required, by the fact $(*)$ of the preceding paragraph applied to $B = K[[x_1, \ldots, x_{d-1}]] \subseteq C_{j_0}$ (C_{j_0} is injectively free over B because L_{j_0} is injectively free over K). $\square$

Lemma. *Let $\mathcal{F}$ be a field, let $\{\mathcal{F}_j\}_{j \in J}$ be a family of subfields, directed by $\supseteq$, with $\bigcap_j \mathcal{F}_j = L$, let K be a subfield of L, and let Φ be a subfield of $\mathcal{F}$ that is a finite purely inseparable algebraic extension of K. Suppose that L and Φ are linearly disjoint[1] in $\mathcal{F}$ over K. Then there is an element $j \in J$ such that $\mathcal{F}_j$ (as well as all smaller members of the family) is linearly disjoint from Φ over K.*

Proof. Fix a basis $w_1, \ldots, w_N$ for Φ over K. As we decrease $\mathcal{F}_j$ the dimension of the vector space V_j spanned by the w_i over $\mathcal{F}_j$ can only increase, but can never be more than N. Therefore, we can choose $\mathcal{F}_j$ so that this dimension is maximum. We want to prove that the w_i are independent over $\mathcal{F}_j$. If not, after renumbering we may suppose that $w_1, \ldots, w_r$ is a basis for V_j with $r < n$. Then w_{r+1} is an $\mathcal{F}_j$-linear combination of $w_1, \ldots, w_r$ in a

[1] That is, elements of a basis for Φ over L remain independent in $L[\Phi]$ (iff $L \otimes_K \Omega \to L[\Phi]$ is an isomorphism, which implies $L \otimes_K \Phi$ is reduced and therefore a field).

unique way. At least one of the coefficients occurring is not in L. Since the family is directed, we can find a smaller field in the family that excludes this coefficient. Over this field, $w_1, \ldots, w_r, w_{r+1}$ are independent, contradicting our choice of $\mathcal{F}_j$. $\square$

Lemma. *Let Ω be a field finitely generated over $L = \operatorname{frac} A$, where $A = K[[x_1, \ldots, x_d]]$ and K is a field of characteristic p. Let Λ be a p-base for K. Then we can choose Γ cofinite in Λ such that $A^\Gamma \otimes_A \Omega$ is a domain.*

Proof. Let $\mathcal{F}_\Gamma = \operatorname{frac} A^\Gamma$. We want to show that we can choose Γ cofinite but sufficiently small that $A^\Gamma \otimes_A \Omega$ is a domain, and since Ω is flat over A, this is contained in $\mathcal{F}_\Gamma \otimes_A \Omega \cong \mathcal{F}_\Gamma \otimes_L \Omega$, and so it will suffice to show that the latter is a domain. Since $\mathcal{F}_\Gamma$ is purely inseparable over L, this will be the case if and only if $\mathcal{F}_\Gamma \otimes_L \Omega$ is reduced. It is clear that if one choice of Γ succeeds then so do all smaller choices. It is also clear that the problem gets harder as Ω gets larger. By replacing Ω by a larger field we may assume without loss of generality that it is obtained from L by first making a purely transcendental extension $L(y) = L(y_1, \ldots, y_s)$ followed by a finite purely inseparable extension followed by a finite separable extension. The purely inseparable part will be contained in $L'(y_1^{1/q}, \ldots, y_s^{1/q})$ for some finite purely inseparable extension L' of L and some q. The separable part may be ignored.[2] We change notation, writing y for $y^{1/q}$, and we may thus assume that $\Omega = L'(y)$ where L' is a finite purely inseparable extension of L. It suffices to show that we can choose Γ so that $L'(y) \otimes_L \mathcal{F}_\Gamma$ is reduced. This ring is a localization of a polynomial ring over $L' \otimes_L \mathcal{F}_\Gamma$, and so it suffices to show that for a fixed finite purely inseparable extension L' of L we can choose Γ cofinite in Λ so that $L' \otimes_L \mathcal{F}_\Gamma$ is reduced, and this is the case iff L' and $\mathcal{F}_\Gamma$ are linearly disjoint over L in L^∞. Since L' and L are linearly disjoint over L and $\cap_\Gamma \mathcal{F}_\Gamma = L$ as Γ runs through the cofinite subsets of Λ by a preceding lemma, it follows from the immediately preceding lemma that for all sufficiently small choices of the cofinite sets Γ, L' and $\mathcal{F}_\Gamma$ are linearly disjoint over L. $\square$

Corollary. *Let R be essentially of finite type over a complete local ring B with coefficient field K of char. $p > 0$, let Λ be a p-base for K, and let $R^\Gamma = B^\Gamma \otimes_B R$ (it is faithfully flat and purely inseparable over R). Then for all sufficiently small Γ cofinite in Λ:*
(a) *if R is a domain, then R^Γ is a domain.*
(b) *if P is prime in R, then PR^Γ is prime in R^Γ.*
(c) *if I is a radical ideal in R, then IR^Γ is a radical ideal in R^Γ.*
(d) *if R is reduced, then R^Γ is reduced.*

Proof. We may replace B by its image in R, and then by a formal power series ring A with the same coefficient field, since $B^\Gamma \cong A^\Gamma \otimes_A B$. Thus, for part (a), it suffices to show that for sufficiently small Γ cofinite in Λ, $A^\Gamma \otimes_A R$ is a domain, and since A^Γ is flat over A, this is contained in $A^\Gamma \otimes_A \operatorname{frac} R$. The result now follows from the preceding Lemma. For (b), apply (a) to R/P, noting that $(R/P)^\Gamma \cong R^\Gamma/PR^\Gamma$. If I is radical, $I = \bigcap_{j=1}^h P_j$, where $P_1, \ldots, P_h$ are its minimal primes. For sufficiently small cofinite Γ, every $P_j R^\Gamma$ is prime, and since R^Γ is flat over R, $IR^\Gamma = \bigcap_j P_j R^\Gamma$. Part (d) is the case where $I = (0)$. $\square$

---

[2] If a reduced ring S contains a field G and G' is purely inseparable over G, then $S' = S \otimes_G G'$ is reduced: since $S \to S'$ is flat, it suffices to show this for the fibers, from which we may assume that $S = \kappa$ is field, while $G' \cong G[z]/\bigl(g(z)\bigr)$ with g square-free, so that $S' \cong \kappa[z]/\bigl(g(z)\bigr)$ with g square-free. $\square$