715: Lecture of November 7, 2002

Proposition. *Let $S \to T$ be a faithfully flat homomorphism of Noetherian rings.*

(a) *If T is regular, then S is regular. (We proved the corresponding facts earlier for the Cohen-Macaulay and Gorenstein properties.)*

(b) *If S contains a field $\mathcal{F}$, $\mathcal{G}$ is a field finitely generated over $\mathcal{F}$, and S is Cohen-Macaulay, Gorenstein, or geometrically regular over $\mathcal{F}$, then $S \otimes_{\mathcal{F}} \mathcal{G}$ is Cohen-Macaulay, Gorenstein, or geometrically regular over $\mathcal{G}$, respectively.*

(c) *If T is weakly F-regular then S is weakly F-regular.*

(d) *If S and T are F-finite and T is strongly F-regular, then S is strongly F-regular.*

(e) *If $c \in S$ is a test element in T, then it is a test element in S.*

(f) *If $c \in S$ is a completely stable test element in T then it is a completely stable test element in S.*

Proof. For (a), the issue is local on S and we may localize at a maximal ideal of S and a prime ideal of T lying over it. Thus, we may assume that $(S, m) \to (T, n)$ is a flat local homomorphism. Choose a minimal free resolution of the residue class field S/m of S, so that the matrices have entries m. Apply $T \otimes_S \_$: since T is flat and m maps into n, this gives a minimal free resolution of T/mT. Since T is regular, this resolution is finite. But then S/mS has a finite projective resolution over S, and so S is regular.

For (b), by induction on the number h of elements needed to generate $\mathcal{G}$ over $\mathcal{F}$, we may reduce to the case where $h = 1$, say $\mathcal{G} = \mathcal{F}[x]/(g)$, where g is a monic polynomial in one variable, or where $\mathcal{G} = \mathcal{F}[x]$. Then $S \otimes_{\mathcal{F}} \mathcal{G} \cong S[x]/(g)$ or to a localization of $S[x]$, and so is Cohen-Macaulay or Gorenstein if S is (note that since the map is flat, this reduces to looking at fibers, and so reduces to the case where S is a field).

We now consider the case where S is geometrically regular over $\mathcal{F}$ and we need to prove that $S \otimes_{\mathcal{F}} \mathcal{G}$ is regular and remains so when we tensor further over $\mathcal{G}$ with a finite purely inseparable extension of $\mathcal{G}$. But that is the same as forming $S \otimes_{\mathcal{F}} \mathcal{G}$ for a larger version of $\mathcal{G}$, and so it suffices to prove regularity. As in the proof of the Lemma on page 3 of the notes for the Lectures of November 4 and 6, the field $\mathcal{G}$ is contained in a finitely generated field extension of $\mathcal{F}$, call it $\mathcal{G}'$, that is obtained in three steps: a finite purely inseparable extension, then a finitely generated pure transcendental extension, and then a finite separable extension. Now $S \otimes_{\mathcal{F}} \mathcal{G}'$ is faithfully flat over $S \otimes_{\mathcal{F}} \mathcal{G}$, and so, by part (a), the fact that the former is regular implies that the latter is regular. Thus, it suffices to prove that $S \otimes_{\mathcal{F}} \mathcal{G}'$ is regular.

The first step produces a regular ring by the definition of geometric regularity over $\mathcal{F}$. The second step produces a ring that is a localization of a polynomial ring over a regular ring, and so is regular. Therefore, working over enlarged versions of S and $\mathcal{F}$, we see that it suffices to consider the case of a finite separable extension, and such an extension is cyclic.

Thus, we may assume that $S \otimes_{\mathcal{F}} \mathcal{G}' = S[x]/(g)$ where $g = g(x)$ is a monic square-free polynomial. Since this is flat over S, we need only show that the fibers are regular. But these have the form $\kappa[x]/(g)$ for some field extension κ of $\mathcal{F}$, and g remains square-free, so that $\kappa[x]/(g)$ is a finite product of finite separable field extensions of κ.

For part (c), if $cu^q \in I^{[q]}$ for all $q \gg 1$ with $c \in S^\circ$, $u \in S$ and $I \subseteq S$, then $c \in T^\circ$, since T is flat over S, and so $u \in (IT)^* \cap S = IT \cap S = I$.

For part (d), suppose that $c \in S^\circ$. Then $c \in T^\circ$, and therefore the map $T \to T^{1/q}$ sending 1 to $c^{1/q}$ splits for $q \gg 1$. Since we have $T \to S^{1/q} \otimes_S T \to T^{1/q}$, it follows that the map $T \to S^{1/q} \otimes_S T$ sending $1 \mapsto c^{1/q} \otimes 1$ splits for $q \gg 1$, so that the map $\mathrm{Hom}_S(S^{1/q}, S) \to \mathrm{Hom}_S(S, S)$ induced by applying $\mathrm{Hom}_S(\_\_, S)$ to the map $S \to S^{1/q}$ sending $1 \mapsto c^{1/q}$ becomes surjective after applying $T \otimes_S \_\_$. Since T is faithfully flat over S, the map was already surjective, as required.

For part (e), note that $c \in T^\circ$ and so $c \in S^\circ$, since any minimal prime of S is contained in a minimal prime of T. Now, if $u \in S$, $I \subseteq S$, and $cu^q \in I^{[q]}$ for all $q \gg 0$, then $u \in (IT)^*$ and so $cu \in IT \cap R = I$.

Finally, for part (f), let P be a prime ideal of S and Q a prime lying over it in T. Then $\widehat{S_P} \to \widehat{T_Q}$ is faithfully flat, and since c is a test element in $\widehat{T_Q}$, it is a test element in $\widehat{S_P}$. $\square$

Corollary. *If $R \to U$ is flat with geometrically regular fibers, and T is essentially of finite type over R, then $T \to T \otimes_R U$ is flat with geometrically regular fibers.*

Proof. Flatness is clear. Let $\mathcal{G}$ be the fraction of T/Q, where Q is prime in T, let Q lie over P in R, and let $\mathcal{F}$ be the fraction field of R/P. Then $\mathcal{G} \otimes_T (T \otimes_R U) \cong \mathcal{G} \otimes_R U \cong \mathcal{G} \otimes_{\mathcal{F}} (\mathcal{F} \otimes_R U)$. But $S = \mathcal{F} \otimes_R U$ is a fiber of $R \to U$, and so is geometrically regular over $\mathcal{F}$, and since $\mathcal{G}$ is finitely generated over $\mathcal{F}$, the result follows from part (b) of the Lemma above. $\square$

Lemma. *Let R be essentially of finite type over a complete local ring B with coefficient field K of characteristic $p > 0$, and let Λ be a p-base for K. Let $R^\Gamma = B^\Gamma \otimes_B R$, which will be faithfully flat and purely inseparable over R. If R is Gorenstein, then R^Γ is Gorenstein.*

Proof. It suffices to prove that $R \to R^\Gamma$ has Gorenstein fibers. We may replace R by R/P for a typical prime P, and so it suffices to show that when R is a domain, $(\mathrm{frac}\, R) \otimes_R R^\Gamma$ is Gorenstein. Suppose that we replace B by a complete regular ring $A \subseteq R$ such that R is essentially of finite type over A. Let $\mathcal{F} \subseteq \mathcal{G}$ be the extension of fraction fields. Then $R^\Gamma \cong A^\Gamma \otimes_A R$, and the fiber may be identified with $A^\Gamma \otimes_A R \otimes_R \mathcal{G} \cong \mathrm{frac}\, A^\Gamma \otimes_{\mathcal{F}} \mathcal{G}$. The result now follows from the proposition above. $\square$

Fact. *If $R \subseteq S$ is purely inseparable, then the induced map $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is a homeomorphism.*

Proof. Every prime of S is the radical of its intersection with R, and the radical of PS is easily checked to be prime for every prime P of R. It follows easily that $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is bijective. If c is in S it may be replaced by a power in R, and the sets of primes containing c correspond under the bijections for any $c \in R$. $\square$

Theorem. *Let R be essentially of finite type over a complete local ring B with coefficient field K of characteristic $p > 0$, and let Λ be a p-base for K. Let $R^\Gamma = B^\Gamma \otimes_B R$, which will be faithfully flat and purely inseparable over R. If R is Gorenstein and weakly F-regular, then for all sufficiently small Γ cofinite in Λ, R^Γ is weakly F-regular and Gorenstein.*

Proof. We already know that R^Γ will be Gorenstein for every choice of Γ. The map of spectra is a homeomorphism no matter what Γ is. Let $X = \operatorname{Spec} R \approx \operatorname{Spec} R^\Gamma$ for all Γ. Let X_Γ denote the locus in X corresponding to the set of primes P in $\operatorname{Spec} R^\Gamma$ such that R^Γ_P is not weakly F-regular. Since R^Γ is F-finite Gorenstein, this is the same as the set of P such that R^Γ_P is not strongly F-regular, and therefore this set is closed. Closed sets in the spectrum of a Noetherian ring have DCC (since ideals have ACC), and therefore we can choose Γ so that X_Γ is minimal. If $\Gamma_0 \subseteq \Gamma$, then $R^{\Gamma_0} \to R^\Gamma$ is faithfully flat, and so $X_{\Gamma_0} \subseteq X_\Gamma$. Therefore, there is a choice of Γ such that X_Γ is minimum, and not just minimal. Choose Γ so as to achieve the minimum for X_Γ. To finish the proof, we shall show that X_Γ is empty. Suppose not, and that $P \in \Gamma$. Then we can choose $\Gamma_1 \subseteq \Gamma$ such that $P_1 = PR^{\Gamma_1}$ is prime. Then $R_P \to (R^{\Gamma_1})_P \cong (R^{\Gamma_1})_{P_1} = S$ is faithfully flat, the rings are Gorenstein, the extension is purely inseparable, and $P_1 S = PS$, so that the maximal ideal of R_P expands to give the maximal ideal of S. By the first Proposition in the notes from the Lectures of November 4 and 6, the fact that R_P is weakly F-regular and Gorenstein implies that S is weakly F-regular and Gorenstein, which means that $P \notin X_{\Gamma_1}$, contradicting that X_Γ is minimum. $\square$

Theorem. *Let R be reduced and essentially of finite type over an excellent local ring, and let $c \in R^\circ$ be such that R_c is regular. Then c has a power that is a completely stable test element.*

Proof. We can assume that R is essentially of finite type over an excellent reduced local ring $B \subseteq R$. Since $\widehat{B}$ is faithfully flat over B, $R_1 = \widehat{B} \otimes_B R$ is faithfully flat over R and also essentially of finite type over $\widehat{B}$. Moreover, the fibers of $R \to R_1$ are geometrically regular by the Corollary to the first Proposition, and so $(R_1)_c$ is still regular. By passing from R, B to R_1, $\widehat{B}$ we may assume that R is reduced and essentially of finite type over a complete local ring B, and we have preserved that R_c is regular. Choose a coefficient field K for B and a p-base Λ for K. Then we can choose Γ cofinite in Λ so small that R^Γ is reduced and $(R_c)^\Gamma \cong (R^\Gamma)_c$ is weakly F-regular and Gorenstein, by the preceding Theorem. Since R^Γ is reduced and F-finite and $(R^\Gamma)_c$ is weakly F-regular and Gorenstein, it is strongly F-regular and Gorenstein, and so c has a power that is a completely stable test element for R^Γ. Since R^Γ is faithfully flat over R, that power is also a completely stable test element for R. $\square$