715: Lecture of November 13, 2002

We have proved that if R is reduced and essentially of finite type over an excellent local ring, then if $c \in R^\circ$ is such that R_c is regular, then c has a power that is a completely stable test element. We next want to prove that the same is true if we make the weaker assumption that R_c is weakly F-regular and Gorenstein instead of regular. The only point at which the proof changes is in showing that the hypothesis is preserved when we pass from R to $\widehat{B} \otimes_B R$. We know that the fibers are geometrically regular, and there is therefore no problem showing that the Gorenstein hypothesis is preserved when we pass from R_c to $(\widehat{B} \otimes_B R)_c$. The issue is to show that weak F-regularity is preserved. Part (d) of the following results suffices, provided that we know that a weakly F-regular Gorenstein ring is F-regular: we shall address this issue next.

Theorem. *Let $R \to S$ be a flat homomorphism of locally excellent Noetherian rings of positive prime characteristic p.*

(a) *If the homomorphism is local, say $(R, m) \to (S, n)$, R is weakly F-regular, the closed fiber S/mS is regular, and the generic fiber $(R - \{0\})^{-1}S$ is regular, then S is weakly F-regular.*

(b) *If the fibers are regular and for every prime P of R lying under a maximal ideal of S, R_P is weakly F-regular, then S is weakly F-regular.*

(c) *If R and S are finitely generated over a field K, R is weakly F-regular, and the fibers are regular, then S is weakly F-regular.*

(d) *If R is F-regular and the fibers are regular then S is F-regular.*

Proof. In order to prove part (a), first note that since R is weakly F-regular it is normal and, hence, approximately Gorenstein. Thus, it has a decreasing sequence of m-primary ideals $\{I_s\}_s$ cofinal with the powers of m. Let d be the dimension of the fiber S/mS, and let $x_1, \ldots, x_d$ be elements of S whose images in S/mS generate n/mS, the maximal ideal of the closed fiber (which is regular). By the Lemma and its Corollary from the notes on the Lecture of October 31, the elements $x_1, \ldots, x_d$ form a regular sequence on S, and on all of the rings S/IS for every ideal I of R; moreover, for every positive integer t, the ring $S/\bigl(IS + (x_1^t, \ldots, x_d^t)S\bigr)$ is faithfully flat over R/I (since the tth powers of the elements $x_1, \ldots, x_d$ also form a regular sequence on S/mS).

We shall show that the ideals $J_{s,t} = I_s + (x_1^t, \ldots, x_d^t)S$, which are clearly n-primary, are irreducible. Obviously, as s and t become large, they are contained in n^h for arbitrarily large h. Thus, it will suffice to prove that these ideals are tightly closed in S. Throughout the rest of the argument, s will be fixed and $I = I_s$ will be a fixed m-primary irreducible ideal in R. It will suffice to show that $J_t = I + (x_1^t, \ldots, x_d^t)S$ is irreducible and tightly closed in S. Since S/IS is flat over R/I, which is a zero-dimensional Gorenstein ring, with regular fiber, we know that S/IS is Gorenstein. Moreover, the images of $x_1, \ldots, x_d$ are a system of parameters in this ring, since mS is nilpotent modulo IS. Therefore, S/J_t is a zero-dimensional Gorenstein ring, which means that J_t is irreducible in S.

It remains to show that every J_t is tightly closed in S. Since R is an excellent domain, we can choose $c_1 \in R - \{0\}$ such that R_{c_1} is regular. Since the generic fiber is regular, and since the regular locus is open in S, we can choose c_2 in $R - \{0\}$ such that S_{c_2} is regular.[1] Thus, we can choose $c = c_1c_2 \in R - \{0\}$ such that R_c and S_c are both regular. After replacing c by a power, we may assume that $c \in R - \{0\}$ is a test element in both R and S.

Now let $u \in R$ generate the socle modulo I. The ring S/J_1 is flat over R/I, and since $n = m + (x_1, \ldots, x_d)S$, the image of m in S/J_1 generates the maximal ideal. The socle in S/J_1 is therefore the same as $\mathrm{Ann}_{S/J_1} m = (\mathrm{Ann}_{R/I} m)S/J_1 = uS/J_1$, so that the image of u in S/J_1 generates the socle in S/J_1. It follows that the image of $u_t = (x_1 \cdots x_d)^{t-1}u$ generates the socle in S/J_t. Suppose that $u_t \in J_t^*$ in S. We shall obtain a contradiction by proving that $u \in I^*$ in R.

From the former condition we have that $cu_t^q \in J_t^{[q]}$ for all $q \gg 1$. Thus,

$$c(x_1 \cdots x_d)^{qt-q}u^q \in I^{[q]}S + (x_1^{qt}, \ldots, x_d^{qt})S$$

for all $q \gg 1$. We think in $S/I^{[q]}S$, which is flat over $R/I^{[q]}R$. Moreover, $x_1, \ldots, x_d$ is a regular sequence on $R/I^{[q]}R$. This implies that in $\overline{S} = S/I^{[q]}S$, we have that

$$(x_1^{qt}, \ldots, x_d^{qt})\overline{S} :_{\overline{S}} (x_1 \cdots x_d)^{qt-q}$$

may be computed formally, and is therefore $(x_1^q, \ldots, x_d^q)\overline{S}$. This means that

$$cu^q \in I^{[q]}S + (x_1^q, \ldots, x_d^q)S$$

for all $q \gg 1$. But we also know from the results quoted at the end of the first paragraph of this proof that $R/I^{[q]} \to S/\bigl(I^{[q]}S + (x_1^q, \ldots, x_d^q)S\bigr)$ is faithfully flat and, consequently, injective. This implies that $cu^q \in I^{[q]}$ for all $q \gg 1$, and so $u \in I^* = I$ since R is weakly F-regular. This is a contradiction, and completes the proof of part (a).

For part (b), note that the condition that S be weakly F-regular is local on the maximal ideals of S, and we may apply part (a) to $R_P \to S_Q$ for every prime ideal P of R lying under a maximal ideal Q of S.

Part (c) follows from (b) because for finitely generated algebras $R \to S$ over K, every maximal ideal Q of S lies over a maximal ideal P of R. (Otherwise, the domain R/P would have a finitely generated extension that is a field. By Noether normalization, after localizing at one element of R/P, this would be a polynomial extension followed by an integral extension, which cannot give a field unless the localization of R/P at one element is a field, and this cannot happen unless R/P is a field.)

Part (d) is immediate from part (a) and the fact that F-regularity is local on the primes of S. $\square$

Our next objective is to show that weakly F-regular Gorentein rings are F-regular.

[1] If the non-regular locus in S is defined by the ideal $\mathfrak{A}$, the multiplicative system $R - \{0\}$ must meet $\mathfrak{A}$, or else a prime Q containing $\mathfrak{A}$ and disjoint from $R - \{0\}$ will be such that S_Q is not regular, and this is also a local ring of $(R - \{0\})^{-1}S$, a contradiction.