715: Lecture of November 14, 2002

Last time we showed that if $R \to S$ is flat, the base R is F-regular, and the fibers are regular then S is F-regular. It is worth noting that the roles of the base and fibers cannot be interchanged: it is *not* true that if the base is regular and the fibers are F-regular then S is F-regular. There is an example[1] due to Anurag Singh of a finitely generated $\mathbb{N}$-graded domain R of dimension 3 over an algebraically closed field K of characteristic $p > 2$ and a form $t \in m - \{0\}$, where m is the homogeneous maximal ideal, such that R/tR is strongly F-regular but R is not weakly F-regular: ideals of R are not even Frobenius closed. Let $P = tK[t]$ and consider the flat map $K[t]_P \to R_m$. The fibers are strongly F-regular but the ring R_m is not weakly F-regular. This example shows that F-regularity does not deform. In this respect, F-regularity differs from such properties as being Cohen-Macaulay, Gorenstein, or regular.

In order to show that weakly F-regular Gorenstein rings are F-regular, we prove:

Theorem. *Let R be a Noetherian ring of characteristic p, let S be a multiplicative system in R, and let I be an ideal of R. Suppose that I is generated by $x_1, \ldots, x_n$, an R-sequence. Then $I^*S^{-1}R = (IS^{-1}R)^*$. In particular, if I tightly closed, then $IS^{-1}R$ is tightly closed.*

Before going on to the preliminary result needed for the proof of this Theorem, we want to discuss the issue of localization in general. Let R, S, and I be as in the first sentence of the Theorem and let $u \in R$. When is true that $u \in (IS^{-1}R)^*$? (Note that every element of $S^{-1}R$ has a unit multiple in the image of R, so that we might as well discuss the localization problem in this form.) The condition is precisely that for all $q \gg 1$ there exists $s_q \in S$ such that $s_q cu^q \in I^{[q]}$. The localization problem is equivalent to replacing the "uncontrolled" sequence of elements $s_q \in S$ by one of the form s^q for $s \in S$. For if we have $s^q cu^q = c(su)^q \in I^{[q]}$ for all $q \gg 1$, then $su \in I^*$ and $u \in I^*S^{-1}R$. In the general case, no one has been able to see how to do this.

For the R-sequence case, we need the following:

Lemma. *Let I be an ideal of an arbitrary Noetherian ring R and let $S \subseteq R$ be a multiplicative system.*
(a) *There is an element $s \in S$ such that $\bigcup_{w \in S} I^m :_R w = I^m :_R s^m$ for all $m \in \mathbb{N}$.*
(b) *If R has prime characteristic $p > 0$ and I is generated by a regular sequence $x_1, \ldots, x_n$, then with s as in part (a), $\bigcup_{w \in W} I^{[q]} : w = I^{[q]} : s^{nq}$ for all q.*

[1] Let K have char. $p > 2$, let $m,\, n > 0$ be integers with m prime to p and $m - m/n > 2$, and let $R = K[A, B, C, D, T]/I = K[a, b, c, d, t]$, where I is the ideal of 2×2 minors of $\begin{pmatrix} A^2 + T^m & B & D \\ C & A^2 & B^n - D \end{pmatrix}$. Let a, b, c, d, and t have weights m, $2m$, $2m$, $2mn$, and 2, resp. Then R/tR is strongly F-regular, while ideals of R are not Frobenius closed.

Proof. Let $T = \mathrm{gr}_I R$. Choose $s \in S$ be such that $\mathrm{Ann}_T s$ is maximal. Then $\mathrm{Ann}_T s = \mathrm{Ann}_T ws$ for all $w \in S$, and $\mathrm{Ann}_T w \subseteq \mathrm{Ann}_T ws = \mathrm{Ann}_T s$. Now suppose w kills u modulo I^m, and $u \in I^r$ but $u \notin I^{r+1}$, $r < m$. Then su is in I^{r+1}. Repeating this argument at most m times shows that $s^m u$ is in I^m.

We now consider (b). Fix q and an element $u \in I^{[q]} : w$ for some $w \in S$. We shall prove by induction on $h \in \mathbb{N}$ that $u_h = s^{q+h}u \in I^{[q]} + I^{q+h}$. This will suffice, for when $h = (n-1)q$ this yields $s^{nq}u \in I^{[q]} + I^{nq} \subseteq I^{[q]}$ when I has n generators. If $h = 0$ this simply says that $s^q u \in I^{[q]} + I^q = I^q$, which follows from the property of s in part (a).

Now suppose that $u_h \in I^{[q]} + I^{q+h}$ for a given $h \geq 0$, say $u_h = \sum_j r'_j x_j^q + \sum_\nu r_\nu x^\nu$ where ν runs through n-tuples of nonnegative integers $(\nu_1, \ldots, \nu_n)$ such that every $\nu_i < q$ and $\sum_j \nu_j = q + h$. Some $w \in S$ multiplies u_h into $I^{[q]}$, giving an equation: $\sum_j r''_j x_j^q = \sum_j wr'_j x_j^q + \sum_\nu wr_\nu x^\nu$, which we can rewrite as $\sum_j a_j x_j^q + \sum_\nu wr_\nu x^\nu = 0$, where $a_j = wr'_j - r''_j$. Since the x_j form a regular sequence and none of the monomials occurring is formally in the ideal generated by the others, each wr_ν must be in I. But then sr_ν is in I for each ν, and so $u_{h+1} = su_h = \sum_j sr'_j x_j^q + \sum_\nu sr_\nu x^\nu$: the first term is in $I^{[q]}$, while the second is in $I^{q+(h+1)}$. $\square$

In part (b) it is not necessary to assume characteristic p nor that q is a prime power, if one agrees that $I^{[q]}$ means $(x_1^q, \ldots, x_n^q)R$. But then $I^{[q]}$ depends on the choice of generators for I, not just on what I itself is.

Proof of the Theorem. We note that $I^* S^{-1}R \subseteq (IS^{-1}R)^*$ for any ideal I. The hard part is to show that $(IS^{-1}R)^* \subseteq I^* S^{-1}R$. If not, choose $u/1$ in $(IS^{-1}R)^* - I^* S^{-1}R$. But $u/1 \in (IS^{-1}R)^*$ means that for some $c \in R^\circ$ and for all $q \gg 1$ we can choose $s_q \in S$ such that $s_q c u^q \in I^{[q]}$. (A priori, we only know that c is not in any minimal prime of R disjoint from S. However, we can modify c by adding to it an element in all the minimal primes of R which meet S and precisely those minimal primes not meeting S to which c does not belong, and then raising the sum to a power. The element we are adding is nilpotent when mapped to $S^{-1}R$.) With s as in the Lemma we then have that $s^{nq}cu^q = c(s^n u)^q \in I^{[q]}$, and then $s^n u \in I^*$, so that $u/1 \in I^* S^{-1}R$. $\square$

Corollary. *Let R be a Gorenstein ring of positive prime characteristic p that is weakly F-regular. Then R is F-regular.*

Proof. It suffices to show that for every prime ideal Q of R, R_Q is weakly F-regular. We may replace R by R_m where m is a maximal ideal of R containing Q. Thus, we may assume that (R, m) is local. Suppose that Q has height h. Then we can choose $x_1, \ldots, x_h$ in Q such that $x_1, \ldots, x_h$ is a system of parameters in R_Q and is part of a system of parameters for R. We know that $(x_1, \ldots, x_h)$ is tightly closed in R, and by the Theorem, it is tightly closed in R_Q. But this implies that R_Q is weakly F-regular, since it is Gorenstein. $\square$

We have now completed the proof of:

Theorem. *Let R be a reduced Noetherian ring of positive prime characteristic p essentially of finite type over an excellent local ring, and let $c \in R^\circ$ be such that R_c is weakly F-regular and Gorenstein. Then c has a power that is a completely stable test element in R.*