715: Lecture of November 18, 2002

Among our next goals are proving that, under mild conditions on a Noetherian domain R of positive characteristic p, the integral closure R^+ of R in an algebraic closure of the fraction field of R is a big Cohen-Macaulay algebra over R. We shall eventually prove this, and then use it to give a criterion for the cycles to be in the tight closure of the boundaries (this phenomenon is called *phantom homology*) in certain complexes. In consequence, we shall be able to prove a vanishing theorem for maps of Tor discussed earlier. We shall also eventually be able to prove Karen Smith's result that tight closure is plus closure for parameter ideals.

We first want to develop criteria for certain finite left complexes to be acyclic, i.e., to have no homology in positive degree. The result that we give generalizes a result of Buchsbaum and Eisenbud, and may be found, although with a different proof, in the book *Finite Free Resolutions* by D. G. Northcott.

Because we shall be working in situations without Noetherian hypotheses, we need to develop a notion of depth suitable for those situations. In particular, both the ring and the module may be non-Noetherian. However, for our purposes here, we need only discuss the case where the ideal is finitely generated.

Let $\mathcal{G}_\bullet$, with differential d, and $\mathcal{K}_\bullet$, with differential d', be two left complexes of R-modules. By the *total tensor product* $\mathcal{T}_\bullet(\mathcal{G}_\bullet \otimes_R \mathcal{K}_\bullet)$ of these two complexes we mean the complex such that the module in degree n is $\bigoplus_{i+j=n} \mathcal{G}_i \otimes_R \mathcal{K}_j$, and such that the differential δ, restricted to $\mathcal{G}_i \otimes_R \mathcal{K}_j$, is determined by the formula $\delta(u \otimes v) = du \otimes v + (-1)^i u \otimes d'v$. The complexes $\mathcal{T}_\bullet(\mathcal{G}_\bullet \otimes_R \mathcal{K}_\bullet)$ and $\mathcal{T}_\bullet(\mathcal{K}_\bullet \otimes_R \mathcal{G}_\bullet)$ are isomorphic. At the module level, use the map η that sends $\mathcal{G}_i \otimes \mathcal{K}_j \cong \mathcal{K}_j \otimes \mathcal{G}_i$ via $u \otimes v \mapsto (-1)^{ij} v \otimes u$. Note that

$$\delta\big(\eta(u \otimes v)\big) = \delta\big((-1)^{ij} v \otimes u\big) = (-1)^{ij}(d'v \otimes u) + (-1)^{ij+j}(v \otimes du),$$

while $\eta\big(\delta(u \otimes v)\big) = \eta(du \otimes v + (-1)^i u \otimes d'v)$ which is

$$(-1)^{(i-1)j} v \otimes du + (-1)^i (-1)^{i(j-1)} d'v \otimes du = (-1)^{ij} d'v \otimes du + (-1)^{ij-j} v \otimes du.$$

Thus, $\delta\big(\eta(u \otimes v)\big) = \eta\big(\delta(u \otimes v)\big)$ since $ij + j \equiv ij - j \bmod 2$.

Given n such complexes $\mathcal{G}_\bullet^{(k)}$ with differentials $d^{(k)}$, we may define the n-fold total tensor product recursively, by the rule

$$\mathcal{T}_\bullet(\mathcal{G}^{(1)} \otimes_R \cdots \otimes_R \mathcal{G}^{(n)}) = \mathcal{T}_\bullet\big(\mathcal{T}_\bullet(\mathcal{G}^{(1)} \otimes_R \cdots \otimes_R \mathcal{G}^{(n-1)}) \otimes_R \mathcal{G}^{(n)}\big).$$

The module one gets in degree k will be

$$\bigoplus_{i_1+\cdots+i_n=k} \mathcal{G}_{i_1}^{(1)} \otimes_R \cdots \otimes_R \mathcal{G}_{i_n}^{(n)},$$

and the value of the differential on an element $u_1 \otimes \cdots \otimes u_n$ such that $u_j \in \mathcal{G}_{i_j}^{(j)}$ (where the sum of the i_j is k) will be

$$\sum_{t=1}^{n} (-1)^{i_1+\cdots+i_{t-1}} u_1 \otimes \cdots \otimes u_{t-1} \otimes d^{(t)}(u_t) \otimes u_{t+1} \otimes \cdots \otimes u_n.$$

From this point on we shall often be less careful with our notation for differentials in complexes, tending to denote all of them by d. This does not cause confusion.

By the *Koszul complex* $\mathcal{K}_\bullet(x;\, R)$ of R with respect to the element x we mean the complex $0 \to R \xrightarrow{x} R \to 0$, where the copy of R on the right occurs in degree 0 and the copy of R on the left in degree 1. By the *Koszul complex* of $\underline{x} = x_1, \ldots, x_n$ on R we mean the total tensor product over R of the n complexes $\mathcal{K}_\bullet(x_k;\, R)$. It is easy to verify that $\mathcal{K}_0(\underline{x};\, R) \cong R \otimes_R \cdots \otimes_R R \cong R$, and we make this identification, and that $\mathcal{K}_k(\underline{x};\, R)$ is free of rank $\binom{n}{k}$ over R (and so vanishes if $k > n$). Moreover, $\mathcal{K}_1(\underline{x};\, R)$ will be free of rank n and have a free basis u_i such that u_i maps to x_i in R under the differential. Let U_i be the element 1 in $\mathcal{K}_1(x_i;\, R)$, and let V_i be the element 1 in $\mathcal{K}_0(x_i;\, R)$. Then u_i may be identified $V_1 \otimes \cdots \otimes V_{i-1} \otimes U_i \otimes V_{i+1} \otimes \cdots \otimes V_n$.

We may identify $\mathcal{K}_k(\underline{x};\, R)$ with $\bigwedge^k \mathcal{K}_1(\underline{x};\, R)$, by letting the generator $W_1 \otimes \cdots \otimes W_n$ such that $W_h = U_h$ for $h = i_1, \ldots, i_k$, where $i_1 < \cdots i_k$, and $W_h = V_h$ otherwise correspond to $u_{i_1} \wedge \cdots \wedge u_{i_k}$. The value of the differential d on $u_{i_1} \wedge \cdots \wedge u_{i_k}$ is then

$$\sum_{t=1}^{k} (-1)^{t-1} x_{i_t} u_{i_1} \wedge u_{i_{t-1}} \wedge u_{i_{t+1}} \wedge \cdots \wedge u_{i_k},$$

and this is true even when there is a repetition among the elements $i_1, \ldots, i_k$, or when they are not necessarily in size order. When α, β are forms in the exterior algebra of degrees h, h' respectively, one has that $d(\alpha \wedge \beta) = d(\alpha) \wedge \beta + (-1)^h \alpha \wedge d(\beta)$.

We can now define the Koszul complex $\mathcal{K}_\bullet(\underline{x};\, M)$ of $x_1, \ldots, x_n$ on an R-module M as $\mathcal{K}_\bullet(\underline{x};\, R) \otimes_R M$. The homology of this complex is denoted $H_\bullet(\underline{x};\, M)$ and is called the *Koszul homology* of M with respect to $\underline{x}$.

Here are some elementary but important observations about Koszul complexes and Koszul homology.

(1) $\mathcal{K}_\bullet(x; M)$ is the complex $0 \to M \xrightarrow{x} M \to 0$.

(2) $\mathcal{K}_\bullet(x_1, x_2; R)$ is the complex

$$0 \to R \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix} x_1 & x_2 \end{pmatrix}} R \to 0,$$

while $\mathcal{K}_\bullet(x_1, x_2; M)$ is the complex

$$0 \to M \xrightarrow{\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}} M^2 \xrightarrow{(x_1 \quad x_2)} M \to 0.$$

(3) The matrices of the maps in the Koszul complex all have entries each of which is either 0, or $\pm x_j$ for some j.

(4) Permuting the x_j gives an isomorphic complex, since total tensor product of complexes is commutative, up to isomorphism.

(5) If $R \to S$ is a ring homomorphism, the x_i in R have images y_i in S, and M is an S-module, viewed also as an R-module via restriction of scalars, then $\mathcal{K}_\bullet(\underline{x}; M) = \mathcal{K}_\bullet(\underline{y}; M)$, because the action of x_i on M is the same as the action of y_i on M for every i.

(6) $H_0(\underline{x}; R) = \mathrm{Coker}\,(R^n \xrightarrow{(x_1 \quad \ldots \quad x_n)} R) = R/(x_1, \ldots, x_n)R$, and, by the right exactness of tensor, more generally, $H_0(\underline{x}; M) = M/(x_1, \ldots, x_n)M$.

(7) $H_n(\underline{x}; M)$ is the kernel of the map $M \to M^n$ that sends u to a vector whose entries are $\pm x_i u$, and so $H_n(\underline{x}; M) = \mathrm{Ann}_M\,(x_1, \ldots, x_n)$.

(8) Given a short exact sequence of modules $0 \to M' \to M \to M'' \to 0$ there is a long exact sequence of Koszul homology:

$$\cdots \to H_{i+1}(\underline{x}; M'') \to H_i(\underline{x}; M') \to H_i(\underline{x}; M) \to H_i(\underline{x}; M'') \to H_{i-1}(\underline{x}; M') \to \cdots$$

which, at the right end, looks like:

$$\cdots \to H_1(\underline{x}; M'') \to H_0(\underline{x}; M') \to H_0(\underline{x}; M) \to H_0(\underline{x}; M'') \to 0$$

One gets this sequence by forming the tensor product over R of $\mathcal{K}_\bullet(\underline{x}; R)$, which is a complex of free R-modules, with the short exact sequence $0 \to M' \to M \to M'' \to 0$. This yields a short exact sequences of complexes, and the snake lemma gives the long exact sequence.

(9) $H_1(\underline{x}; R)$ is the quotient of module of relations $\{(r_1, \ldots, r_n) \in R^n : \sum_j x_j r_j = 0\}$ on $x_1, \ldots, x_n$ with coefficients in R by a module consisting of certain "trivial" relations, which are also referred to as *Koszul relations*: these are the relations spanned by the elements $x_j e_i - x_i e_j \in R^n$, where $e_1, \ldots, e_n$ is the standard basis for R^n. Note that these relations come from the equations $x_j x_i - x_i x_j = 0$. $H_1(\underline{x}; M)$, similarly, is the quotient of the module of relations $\{(u_1, \ldots, u_n) \in M^n : \sum_j x_j u_j = 0\}$ by certain "trivial" or Koszul relations. This time, the trivial relations are spanned by those coming the equations $x_j(x_i u) - x_i(x_j u) = 0$ for some choice of i, j and $u \in M$.

Somewhat deeper is:

Proposition. *Let R be a ring, $x_1, \dots, x_n \in R$, and let M be an R-module. Let $\underline{x}^\circ$ denote the sequence $x_1, \dots, x_{n-1}$. Then there is a short exact sequence of complexes $0 \to \mathcal{K}_\bullet(\underline{x}^\circ; M) \to \mathcal{K}_\bullet(\underline{x}; M) \to \mathcal{K}_{\bullet-1}(\underline{x}^\circ; M) \to 0$ where the subscript $_{\bullet-1}$ in the rightmost nonzero term indicates that degrees are shifted by one. This leads to a long exact sequence of homology:*

$$\cdots \to H_i(\underline{x}^\circ; M) \xrightarrow{\delta_i} H_i(\underline{x}^\circ; M) \to H_i(\underline{x}; M) \to H_{i-1}(\underline{x}^\circ; M) \xrightarrow{\delta_{i-1}} H_{i-1}(\underline{x}^\circ; M) \to \cdots .$$

The connecting homomorphism δ_i is multiplication by $(-1)^i x_n$. In consequence, there are short exact sequences:

$$0 \to H_i(\underline{x}^\circ; M)/x_n H_i(\underline{x}^\circ; M) \to H_i(\underline{x}; M) \to Ann_{H_{i-1}(\underline{x}^\circ; M)}\, x_n \to 0$$

for every i.

Proof. We first construct the short exact sequence of complexes in the case where $M = R$. Since each short row is an exact sequence of free modules, it remains exact when we apply $\_ \otimes_R M$. We take the exterior algebra point of view, so that $\mathcal{K}_\bullet(\underline{x}; R)$ is spanned over R by elements $u_{i_1} \wedge \cdots \wedge u_{i_k}$ with $i_1 < \cdots < i_k$. If one considers the span of those such elements that do not involve u_n, what one gets is a subcomplex that is precisely $\mathcal{K}_\bullet(\underline{x}^\circ; R)$. The quotient is free, and spanned by those $u_{i_1} \wedge \cdots \wedge u_{i_k}$ such that $u_{i_k} = u_n$. When one applies the differential to such a term one gets a sum: one of the terms, in which u_n is dropped and, in effect, is replaced by $\pm x_n$, will be in the subcomplex that we are killing, and may be ignored. The other terms, except for the presence of the "extra" u_n, are the same as in the ordinary Koszul complex $\mathcal{K}_\bullet(\underline{x}^\circ; R)$, except that degrees have been shifted by one.

To understand the connecting homomorphism, consider a cycle z in $\mathcal{K}_i(\underline{x}^\circ; R)$. Pull this back to $\mathcal{K}_{i+1}(\underline{x}; R)$: what we get is $z \wedge u_n$. When we apply the differential, we get $dz \wedge u_n + (-1)^i x_n z = (-1)^i x_n z$, since $dz = 0$, and it is clear that this element may be thought of in $K_i(\underline{x}^\circ; R)$. It is now obvious that $\delta_i(z) = (-1)^i x_n z$.

This set-up is essentially unchanged when we apply $\_ \otimes_R M$.

Note that multiplication by x_n and $-x_n$ on a module have the same kernel and the same cokernel. Getting the short exact sequences described from the long one is now straightforward. $\square$

Corollary. *Let R be a ring, $x_1, \dots, x_n \in R$ and M an R-module.*

(a) *If $x_1, \dots, x_n$ is a possibly improper regular sequence on M, then $H_i(\underline{x}; M) = 0$ for all $i \geq 1$.*

(b) *If A is any ring[1] that maps to R, $X_1, \dots, X_n$ are indeterminates over A, and we extend the A-algebra structure on R to a B-algebra structure, where $B = A[X_1, \dots, X_n]$, by mapping X_j to x_j for every j, then*

$$H_i(\underline{x}; M) = H_i(\underline{X}; M) \cong Tor_i^B(B/(X_1, \dots, X_n)B,\, M).$$

[1] We may always take $A = R$ or $A = \mathbb{Z}$. If there is a base field K such that R is a K-algebra, we may wish to take $A = K$.

(c) *Every* $H_i(\underline{x}; M)$ *is killed by both* $(x_1, \ldots, x_n)R$ *and by* $Ann_R\, M$.

Proof. (a) When $n = 1$ the Koszul complex is $0 \to M \xrightarrow{x_1} M \to 0$, and this case is clear. We use induction on n. If $n \geq 2$ and the result is known of smaller n, the fact that $H_i(\underline{x}; M) = 0$ for $i \geq 2$ is immediate, given the short exact sequences in the preceding Proposition, from the fact that $H_i(\underline{x}^\circ; M)$ and $H_{i-1}(\underline{x}^\circ; M)$ vanish. When $i = 1$ we need to be slightly more careful, and use instead that $H_1(\underline{x}^\circ; M) = 0$ while x_n is not a zerodivisor on $H_0(\underline{x}^\circ; M) = M/(x_1, \ldots, x_{n-1})M$.

We now use part (a) to conclude that $\mathcal{K}(\underline{X}; B)$ is a free resolution of $B/(\underline{X})B$, since the indeterminates are a regular sequence in the polynomial ring B, and so may be used to calculate Tor. Thus, $Tor_i^B(B/(\underline{X})B, M) \cong H_i(\underline{X}; M)$. But since X_i acts the same way that x_i does on M, we have that $\mathcal{K}_\bullet(\underline{X}; M) = \mathcal{K}_\bullet(\underline{x}; M)$, and the result follows.

For part (c), it is obvious that $\mathrm{Ann}_R\, M$ kills the Koszul homology: it kills every module in the Koszul complex. To see that $(x_1, \ldots, x_n)R$ kills the Koszul homology, we use the fact that $(X_1, \ldots, X_n)B$ kills $\mathrm{Tor}_i^B(B/(X_1, \ldots, X_n)B, M)$, since it kills the first module, and that each $x_j - X_j$ kills M, since x_j and X_j act on M in the same way. Thus, the Tor is killed by $(X_1, \ldots, X_n)B + (x_1 - X_1, \ldots, x_n - X_n)B$, and since $x_j = X_j + (x_j - X_j)$, all of the x_j kill the Koszul homology. $\square$