715: Lecture of November 20, 2002

Our next goal is to show that, in the Noetherian case, Koszul homology characterizes depth. We first prove:

Lemma. *Let R be any ring (not necessarily Noetherian), let $x_1, \ldots, x_n \in R$, and let M be any R-module. Suppose that $H_{n-d}(\underline{x}; M) \neq 0$ but $H_j(\underline{x}; M) = 0$ for $j > n-d$. Let $y \in I = (x_1, \ldots, x_d)R$ be a nonzerodivisor on M. Then $H_{n-d+1}(\underline{x}; M/yM) \cong H_{n-d}(\underline{x}; M) \neq 0$, while $H_j(\underline{x}; M/yM) = 0$ for $j > n-d+1$.*

Proof. Since $y \in I$ it kills the Koszul homology modules, and so the long exact sequence for Koszul homology coming from the short exact sequence $0 \to M \xrightarrow{y} M \to M/yM \to 0$ yields:

$$H_j(\underline{x}; M) \xrightarrow{y} H_j(\underline{x}; M) \to H_j(\underline{x}; M/yM) \to H_{j-1}(\underline{x}; M) \xrightarrow{y} H_{j-1}(\underline{x}; M)$$

where the maps induced by multiplication by y are 0. Therefore we get a short exact sequence:

$$0 \to H_j(\underline{x}; M) \to H_j(\underline{x}; M/yM) \to H_{j-1}(\underline{x}; M) \to 0$$

for every j. If both surrounding terms vanish, so does the middle term, which proves that $H_j(\underline{x}; M/yM) = 0$ for $j > n - (d-1)$. If $j = n - d + 1$, the left term vanishes, and so $H_{n-d+1}(\underline{x}; M/yM) \cong H_{n-d}(\underline{x}; M) \neq 0$, and we are done. $\square$

Theorem. *Let $R \to S$ be a homomorphism of Noetherian rings and let $x_1, \ldots, x_n \in R$. Let $I = (x_1, \ldots, x_n)R$. Let M be a finitely generated S-module.*

(a) *$M = IM$ if and only if $IS + Ann_S\, M = S$.*

(b) *The Koszul homology modules $H_i(\underline{x}; M) = 0$ for all i (including $i = 0$) if and only if $M = IM$. If $IM \neq M$, let d be the smallest integer such that $H_{n-d}(\underline{x}; M) \neq 0$. Then d is the same as the length of any maximal regular sequence in I on M: in particular, it follows that all maximal regular sequences in I on M have the same length, which is called the depth of M on I. Moreover, the depth of M on I is the same as the depth of M on IS.*

Proof. (a) If $IS + \mathrm{Ann}M = S$ we can choose $i \in IS$ and $z \in S$ such that $zM = 0$ and $i + z = 1$. Then for all $u \in M$, $u = 1 \cdot u = (i+z)u = iu \in (IS)M = IM$.

Now suppose that $M = IM$ but that $IS + \mathrm{Ann}M$ is a proper ideal of S. Let P be a prime containing $IS + \mathrm{Ann}M$. Then $M_P \neq 0$ because $\mathrm{Ann}_S M \subseteq P$, but $IM = M$ implies that $IM_P = M_P$, and the image of I in S_P is contained in the maximal ideal. This contradicts Nakayama's lemma, and concludes the proof of (a).

(b) If all the Koszul homology vanishes then $M/IM = H^0(\underline{x}; M) = 0$, and so $M = IM$. On the other, if $M = IM$ then $IS + \mathrm{Ann}_S\, M = S$. Since I and $\mathrm{Ann}_S M$ both kill the Koszul homology, it follows that $IS + \mathrm{Ann}_S\, M = S$ kills the Koszul homology.

Now suppose that the Koszul homology does not all vanish and so we have d such that $H_{n-i}(\underline{x};M)=0$ for $i<d$ while $H_{n-d}(\underline{x};M)\neq 0$. We shall show by induction on d that every maximal regular sequence in I on M has length d. If $d=0$ what we must show is that any such sequence has length 0. But then $\operatorname{Ann}_M I\neq 0$, so that there is an element of M that is killed by I, and the result is clear.

Now suppose that $d>0$. We first show that there is an element $y_1=y\in I$ such that y_1 is a nonzerodivisor on M. If not, the the image of every element of I when mapped to S, is a zerodivisor on M. This means that the contractions of the associated primes of M in S to R cover I. This means that I is contained in the contraction of an associated prime of S to R, i.e., that IS is contained in a prime Q of S such that $Q=\operatorname{Ann}_S u$ for some $u\in M$. It follows that I kills u, and this implies that $H_n(\underline{x};M)\neq 0$, a contradiction.

Now suppose that we choose any element $y\in I$ such that y is a nonzerodivisor on M. Any maximal regular sequence begins with such an element y, and the problem of extending y to a maximal regular sequence in I on M is equivalent to the problem of giving a maximal regular sequence in I on M/yM.

By the preceding Lemma, we have that $H_{n-i}(\underline{x};M/yM)=0$ for $i<d-1$ and that $H_{n-(d-1)}(\underline{x};M/yM)\neq 0$. The required result is then immediate from the induction hypothesis. $\square$

In the case where neither the ring R nor the module M need be finitely generated, but the ideal $I=(x_1,\,\dots\,,x_n)R$ is, we want to define the *depth* of M on I as the smallest integer d such that $H_{n-d}(x_1,\,\dots\,,x;M)\neq 0$, or else $+\infty$ if all $H_j(\underline{x};M)$ vanish. To emphasize the lack of finiteness assumptions, we shall use the term *Koszul depth* or K-*depth*. *A priori*, it is not even clear that it is independent of the choice of generators for I, but this is true. We begin by showing:

Proposition-Definition. *The K-depth of M on the finitely generated ideal $I\subseteq R$, as defined just above, is independent of how we choose generators for I.*

Proof. To show this, we compare what is obtained from each of two sets of generators with what is obtained from their union. Thus, it suffices to show that the K-depth does not change when the set of generators of an ideal is enlarged by one element.

If one makes a change of generators in the ideal by applying an invertible $n\times n$ matrix A, one gets an isomorphic Koszul complex. It suffices to set up the isomorphism with $M=R$: one may then apply $\_\otimes_R M$. Suppose that $\big(y_1\ \dots\ y_n\big)=\big(x_1\ \dots\ x_n\big)A$. Using the usual basis elements $u_1,\,\dots\,,u_n$ for $\mathcal{K}_1$ in each Koszul complex, we get a commutative diagram:

$$\begin{array}{ccccc}
\mathcal{K}_1(\underline{x};R) & \xrightarrow{(x_j)} & R & \to & 0\\
{\scriptstyle A}\uparrow & & \uparrow{\scriptstyle 1} & & \\
\mathcal{K}_1(\underline{y};R) & \xrightarrow{(x_j)A} & R & \to & 0
\end{array},$$

and we may extend this to a map of complexes by using

$$\bigwedge^k A:\bigwedge^k\mathcal{K}_1(\underline{y};M)\to\bigwedge^k\mathcal{K}_1(\underline{x};M)$$

at the k th level. To check that the squares commute, we need only verify that

$$\bigwedge^{k-1} A\big(d_y(u_{i_1} \wedge \cdots \wedge u_{i_k})\big) = d_x\big(\bigwedge^k A(u_{i_1} \wedge \cdots \wedge u_{i_k})\big).$$

The left side is

$$\sum_t (-1)^{t-1} y_t (Au_{i_1}) \wedge \cdots \wedge (Au_{i_{t-1}}) \wedge (Au_{i_{t+1}}) \wedge \cdots \wedge Au_{i_k}$$

while the right side is

$$\sum_t (-1)^{t-1} d_x(Au_{i_t})(Au_{i_1}) \wedge \cdots \wedge (Au_{i_{t-1}}) \wedge (Au_{i_{t+1}}) \wedge \cdots \wedge Au_{i_k},$$

so that the result follows because $d_x(Au_{i_t}) = y_t u_{i_t}$, which is the fact, already observed, that the diagram displayed above commutes.

Thus, we can perform elementary column operations on a sequence of elements without changing the isomorphism class of the Koszul complex nor the Koszul homology.

Now, when x_{n+1} is in the ideal $I = (x_1, \ldots, x_n)R$, we can subtract off a sum of multiples of the elements $x_1, \ldots, x_n$ so as to change x_{n+1} to 0. This is an elementary column operation and can be achieved by multiplying by an invertible matrix.

The result that we want now follows from the fact that

$$H_j(\underline{x}, 0; M) \cong H_j(\underline{x}; M) \oplus H_{j-1}(\underline{x}; M).$$

To see this, we go back to the proof of the Proposition from the notes of the lecture of November 18, comparing the Koszul homology arising from a sequence with what is obtained when the last element is omitted. The point is that when the last element is 0, the quotient complex, which is represented by the monomials in the u_j involving u_{n+1} (note that $n+1$ replaces n here) is actually a subcomplex itself, so that the short exact sequence of complexes splits. Thus, the short exact sequences one gets are split, and the isomorphism is immediate from that.

From this it is clear that if all the Koszul homology vanished originally, this remains true, while if the highest nonvanishing Koszul homology module was originally in degree $n-d$, then it is in degree $n+1-d$, as required. $\square$

The following remarks are trivial, and are, of course, equivalent:

(1) The K-depth of M is 0 if and only if some element of M is killed by I.

(2) The K-depth of M is positive if and only if no element of M is killed by I.