715: Lecture of November 21, 2002

It is worth noting that the K-depth of M on I may be finite in the non-Noetherian case even when $IM = M$. E.g., let $R = K[x]$, and $M = K[x, 1/x]/K[x]$. Let $I = xR$. Thus, M is spanned over K by the elements $1/x^s$ for $s = 1, 2, 3, \ldots$, and multiplication by x maps $[1/x^s]$ to $[1/x^{s-1}]$ for $s \geq 2$, while killing $[1/x]$. Then $xM = M$, but the depth of M on I is 0, since xR kills $[1/x]$. The assumption that all of the Koszul homology vanishes is strictly stronger than the assumption that $IM = M$ in the non-Noetherian case.

Another example worth looking at is this. Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring, $n \geq 2$, and let $I = m = (x_1, \ldots, x_n)R$. Let $M = \bigoplus f \in mR/fR$. Clearly, $mM \neq M$, so this module has finite K-depth, at most n. Every element of m is a zerodivisor on M, since $f \in m$ kills the nonzero summand R/fR. But a nonzero element with nonzero components in the summands corresponding to R/f_iR, $1 \leq i \leq k$, will not be killed by any element of m relatively prime to the product $f_1 \cdots f_k$. Therefore, the K-depth of M is positive. In fact, the K-depth of a direct sum, even an infinite one, on I is the infimum of the K-depths of the summands, which in this case will be $n - 1$.

We continue with the discussion of basic properties of K-depth begun last time.

(3) K-depth is unaffected by faithfully flat base change, and cannot decrease with flat base change. The point is that if $\underline{y}$ is the image of $\underline{x}$ in S, and if then $\mathcal{K}_\bullet(\underline{y}; S \otimes_R M) = S \otimes_R \mathcal{K}_\bullet(\underline{x}; M)$, and if S is R-flat it follows that $H_i(\underline{y}; S \otimes_R M) \cong S \otimes H_i(\underline{x}; M)$ for all i. Thus, there can only be greater vanishing of Koszul homology after a flat base change, but which Koszul homology modules vanish is unaffected by a faithfully flat base change. In particular, localization can not decrease K-depth, while adjunction of any family of indeterminates does not change K-depth.

(4) If y is a nonzerodivisor on M in $I = (x_1, \ldots, x_n)R$ then the K-depth of M/yM is precisely one less than the K-depth of M (if either is finite), or else both are $+\infty$. This is immediate from the first Lemma in the notes from the Lecture of November 20.

We next note the following:

Lemma. *Let R be a Noetherian ring and MJ a finitely generated R-module.*

(a) *M has a finite filtration in which each factor is a torsion-free module over R/P_i where P_i is one of the associated primes of M.*

(b) *Let $R \to S$ be a flat homomorphism, where S is also Noetherian. Then $Ass(S \otimes_R M) = \bigcup_{P \in \mathrm{Ass}\, M} Ass(S/PS)$.*

(c) *If $S = R[x_1, \ldots, x_n]$ is a polynomial ring over R, then, with $M[x_1, \ldots, x_n] = S \otimes_R M$, we have that $Ass\, M[x_1, \ldots, x_n] = \{PS : P \in Ass\, M\}$.*

Proof. (a) Let P be a minimal associated prime of M, and let $N \subseteq M$ be the kernel of the map $M \to M_P$. Then $\mathrm{Ass}\, N \subseteq \mathrm{Ass}\, M$, but P does not occur in $\mathrm{Ass}\, N$. By induction on the number of associated primes, it follows that N has such a finite filtration, and it suffices to prove the same for $M/N \subseteq (M/N)_P \cong M_P$. Since the associated primes of M_P may be identified with the expansions of a subset of those of M, it follows that the only associated prime of $(M/N)_P = M_P$ is PR_P, and so the only associated prime for $M/N \subseteq (M/N)_P$ is P. We replace M by M/N. We have reduced to the case where M has just one associated prime P, and so M_P has finite length. We now use induction on the length of M_P. Let V be the annihilator of P in M_P. Let U be the contraction of V to M. Then U is torsion-free over R/P, and M/U embeds in M_P/U. It follows that the only associated prime of M/U is P, and $(M/U)_P$ has smaller length than M_P. By the induction hypothesis, M/U has the required filtration. This concludes the proof of (a).

(b) If $Q \in \mathrm{Ass}\,(S/PS)$ for $P \in \mathrm{Ass}\,M$, then S/Q embeds in S/PS, while applying $S \otimes_R \_$ to the injection $0 \to R/P \to M$ shows, since S is R-flat, that S/PS embeds in $S \otimes_R M$. To prove the other inclusion, we use part (a). Then the associated primes of $S \otimes_R M$ are contained among those of the modules $S \otimes_R T$ as T runs through the various torsion-free factors: suppose that T is torsion-free over R/P, where $P \in \mathrm{Ass}\,M$. Then T embeds in a finitely generated free (R/P)-module, say $(R/P)^{\oplus h}$, and so $S \otimes T$ embeds in $(S/PS)^{\oplus h}$. It follows that $\mathrm{Ass}\,(S \otimes_R T) \subseteq \mathrm{Ass}\,(S/PS)$. This proves (b).

(c) This now follows from (b) because for P prime in R, PS is prime in S. $\square$

We continue with our analysis of the properties of K-depth.

(5) If no element of M is killed by I, then $\sum_{j=1}^{n} x_j t_j$, where the t_j are indeterminates, is a nonzerodivisor[1] on $R[t_1, \ldots, t_n] \otimes_R M = M[t_1, \ldots, t_n]$. If the given element kills some element of M, we can find a finitely generated (over the prime ring) subring of R containing the x_i, call it R_0, and a finitely generated R_0-submodule of M, call it M_0, such that $\sum_{j=1}^{n} x_j t_j$ kills an element of $M_0[t_1, \ldots, t_n]$. By this device we may assume without loss of generality that R and M are Noetherian. But, in that case, the associated primes of $M[t_1, \ldots, t_n]$ over $R[t_1, \ldots, t_n]$ are simply the expansions of those of R to $R[t_1, \ldots, t_n]$. But then we have that $\sum_i x_i t_i$ is in the expansion of an associated prime of R to $R[t_1, \ldots, t_n]$. This implies that all of the x_i are in that associated prime, and, hence, that some element of M is killed by all of the x_i, a contradiction.

(6) It now follows that if the K-depth of M on $I = (x_1, \ldots, x_n)R$ is finite, say d, then after some faithfully flat extension S, and, in fact, after adjoining finitely many indeterminates to R, there is a possibly improper regular sequence of length d in IS on $S \otimes_R M$. Likewise, any possibly improper regular sequence $y_1, \ldots, y_k$ in I on M can be extended to one of length d in IS on $S \otimes_M S$ for some faithfully flat extension S of R, which can be taken to be a polynomial ring. This is a consequence of the fact that the K-depth of $M/(y_1, \ldots, y_k)M$ will be precisely $d - k$. On the other hand, there can be no (possibly improper) regular sequence of length greater than d in IS on $S \otimes_R M$: the faithfully flat extension does not change the K-depth, and killing such a sequence of length n decreases the K-depth of M on IS by n, which is impossible if $n > d$. If the K-depth of R is infinite, every possibly improper regular sequence in I on M can be extended to arbitrarily great

[1] Any polynomial in the t_j such that every x_j occurs as one of the coefficients has the same property: the proof is the same. Thus, we could have adjoined just one indeterminate t and used $\sum_{j=1}^{n} x_j t^j$.

length after the adjunction of finitely many indeterminates (or infinite length after the adjunction of infinitely many indeterminates). The K-depth may be characterized as the supremum of lengths of possibly improper regular sequences in IS on $S \otimes_R M$ as S runs through either finitely generated polynomial rings over R or through all faithfully flat extensions of R, or if S is simply allowed to be a single polynomial ring extension of R, but in infinitely many variables.

(7) If I and J are finitely generated ideals that have the same radical, then the K-depths of M on I and J are equal. Give a polynomial extension ring S of R and a possibly improper regular sequence $f_1, \ldots, f_d$ in IS on $S \otimes_M$, the elements $f_1, \ldots, f_d$ will have powers in JS, so there is a possibly improper regular sequence of the same length in JS. Likewise, for any possibly improper regular sequence in JS, the elements will have powers in the IS. Similarly, if $I \subseteq J$ are finitely generated ideals, then the K-depth of M on I is $\leq$ the K-depth of M on J, and the same holds if I is contained in the radical of J.

(8) If $I = (x_1, \ldots, x_n) \subseteq R$ is finitely generated, $y \in R$, and the K-depth of M on $I + yR$ is finite, then it is either d, or $d+1$, where d is the K-depth of M on I. It is at least d by (7) above. To show that, if finite, it is either d or $d+1$, it suffices to show that $H_j(\underline{x}, y;\, M) = 0$ if $j > n+1-d$. This follows from the short exact sequences from the Proposition of the lecture notes from November 18:

$$0 \to H_j(\underline{x};\, M)/yH_j(\underline{x};\, M) \to H_j(\underline{x}, y;\, M) \to \mathrm{Ann}_{H_{j-1}(\underline{x};\, M)}y \to 0.$$

If $j > n+1-d$, both $H_j(\underline{x};\, M)$ and $H_{j-1}(\underline{x};\, M)$ vanish.