715: Lecture of November 25, 2002

If I is a finitely generated ideal of R and N is any module, the K-depth of N on I is the infimum of the K-depths of N_P on IR_P over R_P as P runs through the prime (or maximal) ideals of R. Since we have already observed that K-depth cannot decrease when we localize, it suffices to show that we can choose P so that localizing at P does not increase the K-depth. But if u is a nonzero element of $H_{n-d}(\underline{x}; N)$ then we can keep this Koszul homology module nonzero by localizing at P containing $\mathrm{Ann}_R\, u$.

We note the following further properties of K-depth, analogous to those of ordinary depth:

Lemma.. *If $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules and I is a finitely generated ideal of R, then:*
(a) *The K-depth of B on I is at least the minimum of the K-depths of A and C on I.*
(b) *If the K-depth of B is strictly larger than the K-depth of C, then the K-depth$_I A = 1+$K-depth$_I C$. If the K-depths of B and C are equal to d, then the K-depth of A on I is at least d.*
(c) *If the K-depth of A on I is at least d and the K-depth of B on I is at least $d-1$, the K-depth of C on I is at least $d-1$.*

Proof. Let $I = (x_1, \dots, x_n)R$. All three proofs are immediate from the long exact sequence for Koszul homology:

$$\cdots \to H_{j+1}(\underline{x};\, C) \to H_j(\underline{x};\, A) \to H_j(\underline{x};\, B) \to H_j(\underline{x};\, C) \to H_{j-1}(\underline{x};\, A) \to \cdots.$$

(a) simply follows because if $H_j(\underline{x};\, A) = 0 = H_j(\underline{x};\, C)$ then $H_j(\underline{x};\, B) = 0$. (b) holds because if $j > n-$K-depth$_I B$, then $H_{j+1}(\underline{x};\, C) \cong H_j(\underline{x};\, A)$, and the result in the first sentence is immediate. The second statement follows because both $H_j(\underline{x};\, B)$ and $H_{j+1}(\underline{x};\, C)$ vanish for $j > n-d$.

For (c), we must show that $H_j(\underline{x};\, C) = 0$ for $j > n-d+1$. But we know that $H_j(\underline{x};\, B) = 0 = H_{j-1}(\underline{x};\, A)$ for j in this range. $\square$

Corollary. *If $0 \to A_h \to A_{h-1} \to \cdots \to A_1 \to A_0 \to B \to 0$ is an exact sequence of R-modules and I is a finitely generated ideal of R such that K-depth$_I A_{h-j} \geq d-j$, $0 \leq j \leq h$, then K-depth$_I B \geq d-h$.*

Proof. If $h = 1$, this is part (c) of the Lemma just above. We use induction on h. If $h > 1$, we may break the exact sequence above into two pieces: let C be the kernel of $A_0 \to B$, and consider $0 \to A_h \to A_{h-1} \to \cdots \to A_1 \to C \to 0$ and $0 \to C \to A_0 \to B \to 0$. Then the K-depth of C on I is at least $d-h+1$ by the induction hypothesis, while the K-depth of B on I is then at least $d-h$ by another application of part (c) of the Lemma above. $\square$

Given a map $\alpha : R^n \to R^m$, the ideal $I_t(\alpha)$ generated by the size t subdeterminants of the matrix does not depend on the choice of free bases for the the two R-modules. To see this note first that the entries of a matrix for $\bigwedge^t(\alpha)$ are the minors. Because $\bigwedge^t(\alpha\beta) = \bigwedge^t(\alpha)\bigwedge^t(\beta)$, we have that $I_t(\alpha\beta) \subseteq I_t(\alpha)$ and also that $I_t(\alpha\beta) \subseteq I_t(\beta)$. It follows that when β is invertible, $I_t(\alpha) = I_t\big((\alpha\beta)\beta^{-1}\big) \subseteq I_t(\alpha\beta) \subseteq I_t(\alpha)$, and we have equality. Similarly, if γ is invertible, $I_t(\alpha) = I_t(\gamma\alpha)$. Our convention is that $I_0(\alpha)$ is the unit ideal: the determinant of a 0×0 matrix is 1 by convention. We define the determinantal rank of α modulo I to be the size of the largest minor of α that is not in I, which is the same as the determinantal rank of the image of the matrix mod I.

The following result was first published by Buchsbaum and Eisenbud in the Journal of Algebra in their paper *What makes a complex exact.* The sufficiency of the condition can also be deduced from an acyclicity lemma of Peskine and Szpiro. The non-Noetherian version was written down by D. G. Northcott in his book *Finite Free Resolutions.* The proof given here is different, and simpler.

Theorem. *Let $G_\bullet$ denote a finite free complex over R, say*

$$0 \to R^{b_n} \to \cdots \to R^{b_1} \to R^{b_0} \to 0.$$

Let $N \neq 0$ be an arbitary R-module. Let α_i denote the matrix of the map $G_i \to G_{i-1}$, $1 \le i \le n$. Let r_i denote the determinantal rank of α_i mod $Ann_R N$. Then $G_\bullet \otimes_R N$ is acyclic if and only if the following two conditions hold:
(1) $b_i = r_i + r_{i-1}$, $1 \le i \le n$.
(2) *The K-depth of N on $I_{r_i}(\alpha_i) \ge i$, $1 \le i \le n$.*

Note: The rank condition (1) says that $r_n = b_n$, that $r_{n-1} = b_{n-1} - b_n$, that $r_{n-2} = b_{n_2} - b_{n-1} + b_n$, and, in general, that $r_{n-t} = b_{n-2} - b_{n-t+1} + \cdots + (-1)^t b_n$.

Proof. In either part we may replace R by $R/\mathrm{Ann}_R N$ without affecting the issues. Therefore, we may assume that N is faithful, and that the determinantal ranks are simply being computed in R. (That N is faithful is not needed in the proof of sufficiency.)

We first prove the sufficiency of (1) and (2). The case where $n = 0$ is vacuous, and we use induction on n. We assume the result for all complexes of length $n - 1$ over all rings.

We first adjoin sufficiently many indeterminates to the ring that $I_{r_i}(\alpha_i)$ contains a possibly improper regular sequence of length at least i for $1 \le i \le n$.

We next show that if the $I_{r_n}(\alpha_n)$ is the unit ideal then the complex is acyclic. If it has homology, then a nonzero element u in one of the higher homology modules will remain nonzero when we localize at a prime that contains $\mathrm{Ann}_R u$. Thus, we will have a local counterexample. (The ideals $I_{r_i}(\alpha_i)$ contain nonzerodivisors, and so cannot become 0 when we localize. Thus, the determinantal ranks don't change, while the depths can only increase.) Since the r_n size minors generate the unit ideal and the ring is quasi-local, one of the minors is a unit. After a change of basis in G_{n-1}, we may assume that the minor coming from the matrix β formed by the first $r_n = b_n$ rows of α_n is invertible. We can change basis in G_n so that the subdeterminant formed from the first r_n rows of α_n is the identity matrix: we want to multiply on the right by the inverse of β, which corresponds to a change of basis in G_n. We can now perform elementary row operations so that the remaining rows of the matrix are all 0. This corresponds to a change of basis in G_{n-1}. Thus, we may assume without loss of generality that in this case, $\alpha_n : G_n \to G_{n-1} \cong G_n \oplus_R G'_{n-1}$ has a splitting. Let $G'_\bullet$ be the shorter complex

$$0 \to G'_{n-1} \to \cdots \to G_0 \to 0.$$

It is easy to check that the hypotheses of the theorem are satisfied by this complex, and so, by the induction hypothesis, it is acyclic when we apply $\_\otimes N$. But then the same is true for $G_\bullet$, since it is the direct sum of $G'_\bullet$ and the complex $0 \to G_n \xrightarrow{=} G_n \to 0$ (concentrated in degrees n and $n-1$), and we still have a direct sum after we apply $\_ \otimes_R N$. This takes care of the case where the ideal of minors of the last matrix is the unit ideal.

Now consider the general case. Let $x_1, \ldots, x_n$ be a possibly improper regular sequence in $I_{r_n}(\alpha_n)$. By the induction hypothesis applied to the truncated complex

$$0 \to G_n \to \cdots \to G_1 \to 0,$$

we find that $H_i(G_\bullet \otimes_R N) = 0$ for $i \geq 2$, and it suffices to prove that $H_1(G_\bullet \otimes_R N) = 0$. Suppose that $u \in H_1(G_\bullet \otimes_R N)$ and $u \neq 0$. By the result of the preceding paragraph, if we localize at any x_i, since $I_{r_n}(\alpha_n)R_{x_i}$ is the unit ideal, the complex becomes acyclic. Therefore, u is killed by a power of every x_i. Replacing the x_i by these powers, we have that u is killed by $J = (x_1, \ldots, x_n)R$. Let Z_1 and B_1 denote the kernel of the map $G_1 \otimes_R N \to G_0 \otimes_R N$ and the image of the map $G_2 \otimes_R N \to G_1 \otimes_R N$ respectively. Then $u \in Z_1/B_1 \subseteq G_1 \otimes_R N/B_1 = C$, and we have an exact sequence

$$0 \to G_n \otimes_R N \to \cdots \to G_2 \otimes_R N \to G_1 \otimes_R N \to C \to 0,$$

where $u \in C$, $u \neq 0$, and u is killed by $x_1, \ldots, x_n$. Now, each $G_j \otimes_R N$ is simply a direct sum of copies of N, and so the K-depth of $G_j \otimes_R N$ on J is n, while the K-depth of C is on J is 0. This contradicts the Corollary just above.

It remains to show the necessity of the condition.