715: Lecture of November 27, 2002

Before we can prove the necessity of the acyclicity criterion stated last time, we need the following result, which does much of the work.

Proposition. *Let $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ be a sequence of finitely generated R-modules such that $\beta\alpha = 0$. Let $N \neq 0$ be an R-module, and suppose that the K-depth of N on $I_r(\alpha)$ is positive and that $I_{r+1}(\alpha)N = 0$ (so that the rank of α modulo $Ann_R N$ is r). Finally, suppose that $F \otimes N \to G \otimes_R N \to H \otimes_R N$ is exact at $G \otimes_R N$. Let $s = b - r$, where b is the rank of G. Then the K-depth of N on $I_s(\beta)$ is positive, and $I_{s+1}(\beta)N = 0$. (Note that if $F = 0$, then $r = 0$, $\alpha = 0$, $I_r(\alpha) = R$, $I_{r+1}(\alpha) = 0$, and the hypotheses that the K-depth of N on $I_r(\alpha)$ be positive and that $I_{r+1}(\alpha)N = 0$ are satisfied.)*

Proof. The situation in not disturbed if adjoin indeterminates. The validity of the hypotheses and conclusion are not disturbed if we localize at a nonzerodivisor x on N that is contained in $I_r(\alpha)$, since $N \subseteq N_x$. We may therefore assume without loss of generality that $I_r(\alpha) = R$. We may also replace R by $R/\mathrm{Ann}_R N$, and so assume that N is faithful and that $I_{r+1}(\alpha) = 0$.

If the result is false, then either there is an element $u \in N$, $u \neq 0$ such that $I_s(\beta)$ kills u, or there is a size $s+1$ minor D of β that does not vanish. In either case, we may localize at a prime that contains the annihilator of u in the first case, or that contains the annihilator of D in the second case. We get a new counter-example in which the ring is quasi-local. We may therefore assume without loss of generality that the ring (R, m, K) is quasi-local. This means that some size r minor of a matrix for α is invertible, while the size $r + 1$ minors vanish. We may assume that the invertible minor occurs in the upper left corner after changes of basis. By further changes of basis we may assume that it is the $r \times r$ identity matrix. We can assume, after performing elementary row operations, that the remaining entries of the first r columns are zero, and, similarly, after performing elementary column operations, that the remaining entries of the first r rows are zero. These also correspond to changes of basis in F and G. Thus, after changes of basis for F and G, the matrix of α has block form $\begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix}$ where the 1 that occurs is an $r \times r$ identity matrix. The condition that the size $r + 1$ minors vanish implies that $C = 0$: if it has a non-zero entry c_{ij} with $i > r$, $j > r$ the minor formed using the first r rows and the i th row and the first r columns and the j th column will have non-zero determinant c_{ij}. Thus, the matrix of α may be assumed to be of the block form $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. This means that the first r elements in the free basis for F map bijectively to the first r elements in the free basis for G, while the other elements in the free basis for F are mapped to 0, and so G is the direct sum of the image F' of F and a module G', where both F' and G' are free.

Let β' be the map $G' \to H$ obtained by restricting β. It is easy to see that we have exactness at $G \otimes_R N$ iff $\beta' \otimes 1_N : G' \otimes_R N \to H \otimes_R N$ is injective. Moreover, G' has rank s, and $I_s(\beta') = I_s(\beta)$.

We have reduced to the case where $F = 0$. We have a map $\beta : G \to H$ of finitely generated free modules such that G has rank s, and such that $G \otimes_R N \to H \otimes_R N$ is injective. We want to prove that the K-depth of N on $I_s(\beta)$ must be positive. Assume, to the contrary that some nonzero element $u \in N$ kills $I_s(\beta)$. We shall show that there is an element in the kernel of $G \otimes_R N \to H \otimes_R N$. We may assume that $G = R^s$, that β has $t \times s$ matrix (a_{ij}), and that all size s minors of the matrix kill $u \in N$, where $u \neq 0$. Replace R by the ring generated R_0 by the a_{ij} over the prime ring, and N by $N_0 = R_0 u$. If there is an element in the kernel working with R_0, N_0, it will also be in the kernel of $G \otimes_R N \to H \otimes_R N$. Thus, for the final step we may assume that R is Noetherian. Moreover, $Ru \cong R/Ann_R\, u$, and in this ring $I_s(\alpha)$ is 0. Thus, it will suffice to show that if R is a Noetherian ring and that all size s minors of the matrix (a_{ij}) vanish, then the map it gives $R^s \to R^t$ cannot be injective. To see this, it suffices to show that there is a kernel after localizing at a minimal prime of R. We may therefore assume that R is an Artin local ring. There are two cases. If all of the a_{ij} are in the maximal ideal, then the matrix will kill any vector whose entries are in the socle of R, a contradiction. On the other hand, if some entry is a unit, then by elementary row and column operations we may assume that the matrix is the direct sum of a 1×1 matrix (1) and an $s - 1 \times t - 1$ matrix α' whose $s - 1$ size minors all vanish. By induction on s, $\operatorname{Ker} \alpha' \neq 0$ and so $\operatorname{Ker} \alpha \neq 0$. □

Proof of the necessity of the acyclicity criterion. Now suppose that $G_\bullet$, a finite complex of finitely generated free modules over R, and $N \neq 0$, a module, are such that $G_\bullet \otimes_R N$ is acyclic. We may replace R by $R/\mathrm{Ann}_R N$. Adjoin an indeterminate to the ring (which will not affect whether the conditions hold). We begin by showing by induction on the length of $G_\bullet$ that the rank conditions $b_i = r_i + r_{i-1}$ hold, and that every ideal of rank size minors has depth at least 1. By the preceding lemma, the injectivity of $\alpha_n : G_n \otimes_R N \to G_{n-1} \otimes_R N$ implies that the rank is b_n, and that the K-depth is at least 1. We next apply the Lemma at the $n - 1$ spot, which proves that $b_{n-1} = r_n + r_{n-1}$ and that the the K-depth of the ideal of rank size minors of α_{n-1} is at least 1. Continuing in this way, repeated application of the Lemma shows, by a straightforward induction, that all of the rank conditions hold and that all the rank size ideals of minors have K-depth at least 1. Because we adjoined an indeterminate, every ideal generated by rank size minors contains a nonzerodivisor. Let y be the product of these, a nonzerodivisor that is in every rank size ideal of minors.

If $n = 1$ we have already proved necessity. We complete the proof by induction on n. If $n > 1$, note that we have a short exact sequence of complexes $0 \to (G_\bullet \otimes_R N) \xrightarrow{y} (G_\bullet \otimes_R N) \to G_\bullet \otimes_R (N/yN) \to 0$. The long exact sequence for homology shows that $H_j\big(G_\bullet \otimes_R (N/yN)\big) = 0$ for $j \geq 2$, and so the complex $(0 \to G_n \to \cdots \to G_1 \to 0) \otimes_R N$ is acyclic. If $N/yN = 0$ then the K-depth of N/yN on each ideal of minors is $+\infty$, and so the K-depth of N on each rank size ideal of minors is $+\infty$, and we are done. We therefore assume that $N/yN \neq 0$. By the induction hypothesis we find that the K-depth of N/yN on $I_{r_i}(\alpha_i)$ is at least $i - 1$ for $i \geq 2$, and this implies that the K-depth of N of $I_{r_i}(\alpha_i) \geq i$ for all $i \geq 2$, while we have already established that the K-depth of N on $I_{r_1}(\alpha_1) \geq 1$. □