715: Lecture of December 2, 2002

If R is any domain, then R^+ denotes the integral closure of R in an algebraic closure of its fraction field. This is a maximal integral extension that is a domain: any other integral extension that is a domain is isomorphic with a subring of it. Note that if $R \subseteq T$ are domains then $T \cong R^+$ over R is and only if T is an integral extension of R and every monic polynomial with coefficients in T factors into monic linear factors.

We also note that if $R \to S$ is any homomorphism of domains then there is a map $R^+ \to S^+$ that extends the map $R \to S$. Because $R \to S$ factors $R \twoheadrightarrow S_0 \subseteq S$ where S_0 is the image of R in S, it suffices to prove this for the case of an injection $R \subseteq S$ and a surjection $R \twoheadrightarrow S$ separately. In the first case, the algebraic closure L_0 of the fraction field of R in an algebraic closure L of the fraction field of S is an algebraic closure of the fraction field of R, and the integral closure of R in L_0 is clearly a subring of the integral closure of S in L. In the second case we may view S as R/P for some prime P. Since R^+ is integral over R, there is a prime Q of R^+ lying over P. Then R^+/Q may be viewed as an extension of R/P, since $Q \cap R = P$. Since R^+/Q is clearly integral over R/P (simply because R^+ is integral over R), and since every monic polynomial in one variable over R^+/Q factors into monic linear factors (lift the monic polynomial to a monic polynomial over R^+ and then take the image of its factorization over R^+), we may identify R^+/Q with $(R/Q)^+$.

We want to prove that if R is an excellent local domain of characteristic $p > 0$, then R^+ is a big Cohen-Macaulay algebra over R: every system of parameters for R is a regular sequence on R^+. This is a very difficult result. Later we shall use this to show that in an excellent local domain R, the tight closure of an ideal generated by part of a system of parameters is the contraction of its expansion to R^+. Whether this is true for all ideals is a major open question.

We define an element c of a local ring R to be a *colon-killer* if for every sequence of elements $x_1, \ldots, x_{k+1}$ that is part of a system of parameters,

$$c\big((x_1, \ldots, x_k) :_R x_{k+1}\big) \subseteq (x_1, \ldots, x_k)R.$$

Such elements are also called *Cohen-Macaulay multipliers* in the literature.

The proof involves several steps.

(1) Show the existence of plentiful colon-killers.

(2) Show the existence of systems of parameters with "good" colon-killing properties, which are called *extraordinary systems of parameters.*

(3) Prove an "equational lemma" useful for showing that in certain cases, one can force an element in $(x_1, \dots, x_k) :_R x_{k+1}$ into $(x_1, \dots, x_k)S$ in a suitable module-finite extension domain S.

(4) Prove that the lemma in (3) suffices to get rid of relations on extraordinary parameters.

(5) Prove a "moving lemma" that shows that it suffices to get rid of relations on extraordinary parameters.

(6) Assemble these ingredients into a proof that R^+ is a big C-M algebra.

The first of the several important results that we shall need is this:

Theorem (existence of plentiful colon-killers). *Let R be an excellent local ring such that R is equi-dimensional, i.e., for every minimal prime P of R, $\dim R/P = \dim R$. Let $c \in R$ be such that R_c is Cohen-Macaulay. Then c has a power that is a colon-killer.*

Before we can start on the proof of this, we need several preliminary results.

Lemma (Ext characterization of depth). *Let $R \to S$ be a homomorphism of Noetherian rings. Let N be a finitely generated R-module and let M be a finitely generated S-module. Let $I = \mathrm{Ann}_R N$. Then all of the modules $\mathrm{Ext}^j_R(N, M)$ vanish if and only if $IM = M$. If $IM \neq M$ then the smallest value of j such that $\mathrm{Ext}^j_R(N, M) \neq 0$ is the depth d of M on I.*

Proof. We showed earlier that $IM = M$ iff $IS + \mathrm{Ann}_S M = S$. Since I kills N, it kills $\mathrm{Ext}^j_R(N, M)$, and since this is an S-module via the S-module structure on M, we have that IS kills $\mathrm{Ext}^j_R(N, M)$. Likewise, $\mathrm{Ann}_S M$ kills $\mathrm{Ext}^j_R(N, M)$, and so the sum, which we are assuming is S, kills all of these Ext modules.

We now assume that $IM \neq M$ and we use induction on the depth d of M on I. If $d = 0$, we want to show that $\mathrm{Hom}_S(N, M) \neq 0$. In this case, the union of the contractions of the associated primes of M to R contains I, and so I is contained in one of them. Thus, some element $u \in M - \{0\}$ has annihilator precisely Q, where Q is a prime ideal of S and I maps into Q. It will suffice to show that the Hom is non-zero after we localize at the contraction P of Q to R. Replacing R, S, N, M by their localizations at P, we may assume that (R, P) is local and that N is a non-zero R-module (since $I \subseteq P$), while $u \in M$ is killed by P. It suffices to give a non-zero map of N to $Ru \subseteq M$, where $Ru \cong R/P$. By Nakayama's lemma, N/PN is a non-zero vector space over the field R/P, and so we have surjections $N \twoheadrightarrow N/NP \twoheadrightarrow R/P$, and we are done.

We now assume that $d > 0$. Let $x \in I$ be a nonzerodivisor on M. Then M/xM has depth $d-1$ on I, and so, by induction, we have that $\mathrm{Ext}^j_R(N, M/xM)$ vanishes for $j < d-1$, while $\mathrm{Ext}^{d-1}_R(N, M/xM) \neq 0$. The short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$ yields a long exact sequence for Ext in which the maps $\mathrm{Ext}^j_R(N, M) \xrightarrow{x} \mathrm{Ext}^j_R(N, M)$ are induced by multiplication by x, and so are 0, since x kills N. This long exact sequence therefore breaks up into short exact sequences:

$$0 \to \mathrm{Ext}^j_R(N, M) \to \mathrm{Ext}^j_R(N, M/xM) \to \mathrm{Ext}^{j+1}_R(N, M) \to 0$$

Thus, $\mathrm{Ext}^j_R(N,\,M)=0$ for $j\le d-1$, since $\mathrm{Ext}^j_R(N,\,M/xM)$ maps onto $\mathrm{Ext}^{j+1}_R(N,\,M)$, and then taking $j=d-1$ shows that $0\neq \mathrm{Ext}^{d-1}_R(N,\,M/xM)\cong \mathrm{Ext}^d_R(N,\,M)$. $\square$

Lemma (Ext characterization of projective dimension). *Let M be a finitely generated nonzero module over the regular local ring (R,m,K). Let h be the length of a minimal free resolution of M by finitely generated free R-modules. Then h is the largest integer such that $Ext^h_R(M,\,R)\neq 0$. (h is called the projective dimension of M as an R-module.) In fact, the least number of generators of $Ext^h_R(M,\,R)$ is the rank b_h of the last free module in the minimal free resolution.*

Proof. We can use the minimal resolution to compute Ext, and so it is clear that the h is at most the length of the resolution. Call the tail end of the resolution

$$0\to R^{b_h}\xrightarrow{\alpha} R^{b_{h-1}}\to\cdots.$$

The minimality of the resolution implies that α has entries in m. But then $\mathrm{Ext}^h_R(M,\,R)$ is the cokernel of the transpose of the matrix α. Since this matrix has all entries in m, if we tensor with $R/m=K$, the cokernel becomes $\cong K^{b_h}\neq 0$, which proves the final statement and the fact that the Ext is not 0. $\square$

We recall:

The Auslander-Buchsbaum theorem. *Let M be a non-zero module of finite projective dimension over a local ring (R,m,K). Then the projective dimension of M over R is $depth_m R - depth_m M$.*

This is left as an exercise for those who have not seen it before. Next:

Lemma (Ext characterization of the Cohen-Macaulay property). . *Let T be regular local, and $I\subseteq T$ be an ideal of T such that all minimal primes of I have height h. Then T/I is Cohen-Macaulay if and only if all $Ext^j_T(T/I,T)$ vanish for $j>h$. (This leaves a unique non-vanishing Ext, namely $Ext^h(T/I,\,T)$.)*

Proof. The dimension of T/I is $n-h$, where n is the dimension of T. By the Auslander-Buchsbaum theorem, it will be Cohen-Macaulay, i.e., have depth $n-h$ (this is equivalent to depth $\geq n-h$, since the depth cannot exceed the dimension of the local ring) iff its projective dimension is h, but we may also say iff its projective dimension is at most h. By the Ext characterization of projective dimension, this is simply the statement that $\mathrm{Ext}^j_T(T/I,\,T)=0$ for $j>h$. By the Ext characterization of depth we also have that $Ext^j_T(T/I,\,T)$ will then vanish for $j<\mathrm{depth}_I T$, and since T is regular, hence, Cohen-Macaulay, this is the same as h , the height of I. $\square$

Lemma (Ext duals for finite length modules over a regular ring). *Let (T,m,K) be regular of dimension n. Then $Ext^n_T(\_\_,\,T)$ is a contravariant exact functor from modules of finite length over T to themselves. The Ext dual of K is $\cong K$, and this functor preserves length. If $N^\vee$ denotes $Ext^n_T(N,\,T)$, then $(N^\vee)^\vee\cong N$, and N and $N^\vee$ have the same annihilator.*

Proof. Since the annihilator of N in T will be a power of m (or the unit ideal), we have that $\mathrm{Ext}^j_T(N,T)=0$ for $j<n$, and so the fact that this is a contravariant exact functor

is immediate from the long exact sequence for Ext. Let $x_1, \dots, x_n$ be minimal generators for m. Then $\mathcal{K}_\bullet(\underline{x}; T)$ is a free resolution of $T/(x_1, \dots, x_n)T = K$, and we can use it to compute $K^\vee = \mathrm{Ext}^n_T(K, T)$. Since the tail end of the Koszul complex is given by a map $T \to T^n$ such that the entries of the $n \times 1$ matrix are $\pm x_i$, it follows that $\mathrm{Ext}^n_T(K, T) \cong K$. Since the Ext dual is exact, the dual of module having a filtration in which there are s factors, each isomorphic with K, will be a module having a filtration with s factors, each isomorphic to $K^\vee \cong K$. Thus the Ext dual preserves length.

Now suppose that

$$0 \to T^{b_n} \xrightarrow{\alpha_n} T^{b_{n_1}} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_2} T^{b_1} \xrightarrow{\alpha_1} T^{b_0} \to 0$$

gives a minimal free resolution of N. Then, since all the $\mathrm{Ext}^j_T(N, T)$ vanish for $j \neq n$, we have that

$$0 \leftarrow (T^{b_n})^* \xleftarrow{\alpha_n^*} (T^{b_{n_1}})^* \xleftarrow{\alpha_{n-1}^*} \cdots \xleftarrow{\alpha_2^*} (T^{b_1})^* \xleftarrow{\alpha_1^*} (T^{b_0})^* \leftarrow 0,$$

where * indicates $\mathrm{Hom}_T(\_, T)$, is exact except at the left end, where the cokernel is precisely $N^\vee$. Thus, the free complex displayed just above is a free resolution of $N^\vee$, which is evidently killed by $\mathrm{Ann}_T N$ and so has finite length. It is now obvious that using this complex to calculate $(N^\vee)^\vee$ produces a resolution isomorphic to the original resolution of N, since $\mathrm{Hom}_T\big(\mathrm{Hom}_T(\_, T), T\big)$ is isomorphic to the identity functor on the category of finitely generated free T-modules. Since $\mathrm{Ann}_T N$ obviously annihilates $N^\vee$, it follows that $\mathrm{Ann}_T N^\vee$ annihilates $(N^\vee)^\vee \cong N$, and so the two annihilators must be equal. $\square$