715: Lecture of December 9, 2002

Proof of the Theorem on the existence of plentiful colon-killers. Since $R \to \widehat{R}$ is flat with geometrically regular fibers (the latter condition holds because R is excellent), the same is true for the map $R_c \to \widehat{R}_c$: the fibers are a subset of those of the map $R \to \widehat{R}$. Since the base is given to be Cohen-Macaulay, we see that $\widehat{R}_c$ is Cohen-Macaulay as well. Thus, it suffices to prove the theorem for $\widehat{R}$, because any system of parameters for R is a system of parameters in $\widehat{R}$, and if we know that $x_1, \ldots, x_{k+1}, c \in R$ are such that

$$c^N\big((x_1, \ldots, x_k) :_{\widehat{R}} x_{k+1}\big) \subseteq (x_1, \ldots, x_k)\widehat{R}$$

then

$$c^N\big((x_1, \ldots, x_k) :_R x_{k+1}\big) \subseteq (x_1, \ldots, x_k)\widehat{R} \cap R,$$

and this last is $(x_1, \ldots, x_k)R$ because $R \to \widehat{R}$ is faithfully flat.

We have therefore reduced to the case where R is complete, and so we may write $R = T/I$, where T is a complete regular local ring, say of dimension n, and I is an ideal of T all of whose minimal primes have the same height, say h (since R is equidimensional). Note, however, that we are allowing R to have embedded primes. Then $n - h = d = \dim R$.

The fact that R_c is Cohen-Macaulay means that for all primes Q of T such that $I \subseteq Q$ and $c \notin Q/I$, we have that $\mathrm{Ext}^j_{T_Q}(T_Q/IT_Q,\, T_Q) = 0$ for $j > h$, by the Ext characterization of the Cohen-Macaulay property for quotients of regular local rings given last time, and this means that $\mathrm{Ext}^j_T(T/I,\, T)_Q = 0$ for all primes Q such that $I \subseteq Q$, $c \notin Q/I$, and $j > h$. We claim that $\mathrm{Ext}^j_T(T/I,\, T)_c = 0$ for all $j > h$: if not, we can choose a prime of $(T/I)_c$ in its support, which must correspond to a prime Q of T such that $I \subseteq Q$ and $c \notin Q/I$, a contradiction. Since each $\mathrm{Ext}^j_T(T/I,\, T)_c = 0$, $h < j \leq n$ (all the Ext modules over T vanish for $j > n = \dim T$), we can choose a power of c that kills all of the modules $\mathrm{Ext}^j_T(T/I,\, T)$ for $j > h$. We change notation and replace c by this power. Thus, from now on, we assume that c kills every $\mathrm{Ext}^j_T(T/I,\, T)$ for $j > h$.

It suffices to show by induction on $d = \dim R$ that if c kills all $\mathrm{Ext}^j_T(T/I, T)$ then for all $x_1, \ldots, x_k$ that are part of a system of parameters, $c^{E(d)}$ kills all $H_j(\underline{x}; R)$, $j \geq 1$, where $E(d) = 2^d - 1$. Note that $E(d) - 1 = 2E(d-1)$ for $d \geq 1$. In fact, even knowing this for $j = 1$, we see that if $ux_{k+1} = \sum_{t=1}^k u_t x_t$ then $(-u_1, \ldots, -u_k,\, u)$ is a relation on $x_1, \ldots, x_{k+1}$, and so, with $C = c^{E(d)}$, $(-Cu_1, \ldots, -Cu_k,\, Cu)$ is a trivial Koszul relation. Since the generators of the trivial Koszul relations have either 0 or $\pm x_j$ for $j < k+1$ in the last spot, this implies that $Cu \in (x_1, \ldots, x_k)R$, as required.

We now carry through the needed induction. There is nothing to prove for $d = 0$, since any system of parameters is empty in that case. Now suppose that $d \geq 1$. Thus, $n > h$.

Let $N = \bigcup_t \mathrm{Ann}_R m^t$, the ideal of all elements of R killed by a power of the maximal ideal of R. Let $S = R/N$. The ideal N consists of nilpotents, and so $\dim S = \dim R = d$. Since N has finite length over R, it has finite length over T and is killed by a power of the maximal ideal of T. It follows that $\mathrm{Ext}^j_T(N,\,T) = 0$ for $j < n$ (since the depth of T on the annihilator of N is n), while $\mathrm{Ext}^n_T(N,\,T) = N^\vee$, the Ext dual of N over T. From the long exact sequence for Ext coming from the short exact sequence $0 \to N \to R \to S \to 0$, we see that

$$0 \to \mathrm{Ext}^n_T(S,\,T) \to \mathrm{Ext}^n_T(R,\,T) \to \mathrm{Ext}^n_T(N,\,T) \to 0$$

is exact while $\mathrm{Ext}^j_T(S,\,T) \cong \mathrm{Ext}^j_T(R,\,T)$ for $j < n$. Since c kills $\mathrm{Ext}^n_T(R,\,T)$, we have that c kills $N^\vee$, and so c kills the Ext dual of $N^\vee$, which is N. We also have that c kills $\mathrm{Ext}^j_T(S,\,T)$ for $j > h$, and so S satisfies the same hypothesis that R does. But S has depth at least 1.

It will suffice to show that if $d \geq 1$, S is an equidimensional quotient of T of dimension d and of positive depth, and c kills every $\mathrm{Ext}^j_T(S,\,T)$ for $j > h = n - d$, then $c^{E(d)-1}$ kills all the $H_j(x_1,\,\ldots\,,x_k;S)$ for every choice of $x_1,\,\ldots\,,x_k$ that is part of a system of parameters for R (this is equivalent to the assertion that their images form part of a system of parameters in S). The reason that this is sufficient is that from the short exact sequence $0 \to N \to R \to S \to 0$ we have a long exact sequence of Koszul homology

$$\cdots \to H_j(x_1,\,\ldots\,,x_k;\,N) \to H_j(x_1,\,\ldots\,,x_k;\,R) \to H_j(x_1,\,\ldots\,,x_k;\,S) \to \cdots.$$

In a sequence of three modules that is exact at the middle spot, if a kills the first module and b kills the third middle, then ab kills the module at the middle spot. Since c kills N it kills all the $H_j(x_1,\,\ldots\,,x_k;\,N)$. Thus, if we also know that $c^{E(d)-1}$ kills $H_j(x_1,\,\ldots\,,x_k;\,S)$ for $j \geq 1$, we get that $c^{E(d)}$ kills $H_j(x_1,\,\ldots\,,x_k;\,R)$ for $j \geq 1$.

To complete the last step of proving that $c^{E(d)-1}$ kills the modules $H_j(x_1,\,\ldots\,,x_k;\,S)$ for all $j \geq 1$, we consider two cases. If $k < d$, avoid the minimal primes of $(x_1,\,\ldots\,,x_k)S$ and the associated primes of 0 in S (S has positive depth): choose $y \in m_S$ not in any of these. In this case, $x_1,\,\ldots\,,x_k,\,y$ is still part of a system of parameters and y is a nonzerodivisor in the ring.

If $k = d$, avoid $m_S(x_1,\,\ldots\,,x_d)$ and the minimal primes of 0 in S to give a minimal generator y of $(x_1,\,\ldots\,,x_d)S$ that is not a zerodivisor. Extend this to a new set of d minimal generators of $(x_1,\,\ldots\,,x_d)S$. The new geenerators come from the original ones by the action of an invertible $d \times d$ matrix over R, and so the Koszul complex of the new generators is isomorphic with the original Koszul complex and the Koszul homology does not change. Thus, in the second case we may assume that the system of parameters has the form $x_1,\,\ldots\,,x_{d-1},\,y$, where $x_d = y$ is not a zerodivisor.

Let $z = y^t$. Then $0 \to S \xrightarrow{z} S \to S/zS \to 0$ yields a long exact sequence for Ext:

$$\cdots \to \mathrm{Ext}^{j-1}_T(S,\,T) \to \mathrm{Ext}^j_T(S/zS,\,T) \to \mathrm{Ext}^j_T(S,T) \to \cdots$$

Since c kills the left term and the right term for $j > h+1$ we find that c^2 kills the middle term for $j > h+1$. We may therefore apply the induction hypothesis to S/zS to conclude that $(c^2)^{E(d-1)} = c^{E(d)-1}$ kills $H_j(x_1,\,\ldots\,,x_k;S/y^tS)$ for all positive integers t.

In the first case, where $k < d$, if γ represents a cycle in $\mathcal{K}_j = \mathcal{K}_j(x_1, \ldots, x_k; S)$ then $c^{E(d)-1}\gamma$ becomes a boundary if we apply $\_ \otimes_S R/zS$, which places it in $B_j + y^t\mathcal{K}_j$ for all t, where B_j is the image of $\mathcal{K}_{j+1}$ in $\mathcal{K}_j$. Since B_j is m_S-adically closed in $\mathcal{K}_j$, we find that $c^{E(d)-1}\gamma \in B_j$, which shows that $c^{E(d)-1}$ kills $H_j(x_1, \ldots, x_k;\, S)$ for $j \geq 1$, as required.

The remaining case is where $k = d$, $x_1, \ldots, x_d$ is a full system of parameters, and $x_d = y$ is a nonzerodivisor on S. In this case, $H_j(x_1, \ldots, x_{d-1}, y;\, S) \cong H_j(x_1, \ldots, x_{d-1};\, S/yS)$ (which we leave as an exercise), and the result now follows from the fact that for $j \geq 1$, $(c^2)^{E(d-1)} = c^{E(d)-1}$ kills $H_j(x_1, \ldots, x_{d-1};\, S/yS)$ for $j \geq 1$ by the induction hypothesis. $\square$

A ring R is called *biequidimensional* if for every maximal ideal m and minimal prime P of R contained in m, $\dim R_m/PR_m = \dim R$. This is equivalent to requiring that all local rings at maximal ideals be equidimensional of the same dimension. In a biequidimensional semilocal ring R, we call a sequence $x_1, \ldots, x_d$ a system of parameters if it is in the Jacobson radical and is a system of parameters in every local ring of R at a maximal ideal.

Corollary. *Let R be a biequidimensional excellent semilocal ring and c an element such that R_c is Cohen-Macaulay. Then c has a power c^N such that for every system of parameters $x_1, \ldots, x_d$ of R and for all k, $0 \leq k \leq d-1$,*

$$c^N\big((x_1, \ldots, x_k)R :_R x_{k+1}\big) \subseteq (x_1, \ldots, x_k)R.$$

Proof. We can choose such a power for each of the finitely many localizations R_m of R at a maximal ideal, and then take the maximum of the exponents used. The result follows because the inclusion can be checked locally at maximal ideals, and the colon operation commutes with localization. $\square$

We next prove a result that will enable us to force elements into the expansions of ideals to module-finite extension rings.

Theorem (equational forcing lemma). *Let R be a ring of prime characteristic $p > 0$, let $x_1, \ldots, x_k$ be nonzerodivisors in R, and let $J = (z_1, \ldots, z_s)R$ be an ideal such that $J^{[p]} \subseteq I^{[p]} + (x_1 \cdots x_k)^{p-1}J$. Then there is a module-finite extension S of R such that $J \subseteq IS$. Moreover, if R is a domain, then S can be chosen to be a domain.*

Before giving the proof, we indicate how one may be led to the idea for a proof. Let $x = x_1 \cdots x_k$. For every z_i, $1 \leq i \leq s$, we have an equation

$$(*_i) \qquad z_i^p = \sum_{h=1}^{k} r_{ih}x_h^p + x^{p-1}\sum_{j=1}^{s} r'_{ij}z_j.$$

Let U_{ih}, $1 \leq i \leq s$, $1 \leq h \leq k$, be indeterminates. We want to choose values u_{ij} for these in a module-finite extension S of R such that

$$z_i = \sum_{h=1}^{k} u_{ih}x_h$$

for every i, $1 \leq i \leq s$. Substituting for the z_i and z_h in the equations $(**_i)$, we see that we are going to want that the equations

$$(\sum_{h=1}^{k} U_{ih} x_h)^p = \sum_{h=1}^{k} r_{ih} x_h^p + x^{p-1} \sum_{j=1}^{s} r'_{ij} (\sum_{h=1}^{k} U_{jh} x_h)$$

hold. We rewrite this equation as

$$(**_i) \qquad \sum_{h=1}^{k} U_{ih}^p x_h^p = \sum_{h=1}^{k} r_{ih} x_h^p + \sum_{h=1}^{k} \sum_{j=1}^{s} r'_{ij} U_{jh} x^{p-1} x_h.$$

Let y_h be the product of the x_ν^{p-1} for $\nu \neq h$, so that $x_h^{p-1} y_h = x^{p-1}$. The equations $(**_i)$ will follow if we know that for all i, h we have that the equations

$$(**_{ih}) \qquad U_{ih}^p = r_{ih} + \sum_{j=1}^{s} r'_{ij} y_h U_{jh}$$

hold, for we may then multiply $(**_{ih})$ by x_h^p and sum over h, $1 \leq h \leq k$, to recover the equations $(**_i)$.

This suggests what to do: we want to show that we can form a module-finite extension in which the U_{ih} can be given values u_{ih} that make these last equations hold. We then get elements $z'_i = \sum_{h=1}^{k} u_{ih} x_h$ which satisfy the same equations as the z_i. Finally, we shall show that the $z_i - z'_i$ can be forced into $(x_1, \ldots, x_k)S'$ in a further module-finite extension.

We need a preliminary result first.