715: Lecture of December 12, 2002

We define the *unmixed* part I^{unmx} of an ideal I to be the intersection of those primary components of I corresponding to minimal primes of I. We shall exclusively be applying this notion in the case where R is an unmixed (i.e., there are no embedded primes of (0) in R) bi-equidimensional excellent semilocal ring and $I = (x_1, \ldots, x_k)R$, where $x_1, \ldots, x_k$ is part of a system of parameters in the Jacobson radical of R. It is then not difficult to see that R/I^{unmx} has the same properties that R had, except that the Krull dimension is now $d-k$.

We shall refer to R as *standard* if it is a bi-equidimensional excellent unmixed semilocal ring.

By a *sequence of parameters* in a bi-equidimensional semilocal ring R we mean part of a system of parameters in the Jacobson radical of R.

We shall say that a sequence of parameters $x, x_1, \ldots, x_k$ is *extraordinary* if x is a colon-killer, and the image of each x_j, $1 \le j \le k$ is a colon-killer in the ring $R/(x_1, \ldots, x_{i-1})^{\mathrm{unmx}}$. We shall eventually show that there are abundantly many extraordinary systems of parameters in a certain technical sense.

If I is an ideal of R and $x \in R$ we write $I :_R x^\infty$ for $\bigcup_t I_R x^t$, which is also the contraction of IR_x to R. Note that if x is a colon-killer and $x, x_1, \ldots, x_k$ is a sequence of parameters, then for any positive integers $a_1, \ldots, a_k$, $(x_1^{a_1}, \ldots, x_k^{a_k})R :_R x^\infty = (x_1^{a_1}, \ldots, x_k^{a_k})R :_R x$.

The following two results are critical in showing that R^+ is a big Cohen-Macaulay algebra over the excellent local domain R in prime characteristic $p > 0$.

Theorem (unraveling for extraordinary parameters). *Let R be standard and let $x, x_1, \ldots, x_k$ be an extraordinary sequence of parameters. Let $a_1, \ldots, a_k$ be arbitrary positive integers. Then:*

$$(x_1^{a_1}, \ldots, x_k^{a_k})R :_R x^\infty = \sum_{i=0}^{k-1} \Big(\prod_{\nu=1}^{i} x_\nu^{a_\nu - 1}\Big)\big((x_1, \ldots x_i, x_{i+2}^{a_{i+2}}, \ldots, x_n^{a_n})R :_R x^\infty\big)$$

$$+\Big(\prod_{\nu=1}^{k} x_\nu^{a_\nu - 1}\Big)\big((x_1, \ldots, x_k)R :_R x^\infty\big).$$

We postpone the proof. Using this, we shall show:

Critical Lemma. *Let R be a standard domain and let $x, x_1, \ldots, x_k$ be an extraordinary sequence of parameters. Let $S = R^+$. Let $I = (x_1, \ldots, x_k)R :_R x^\infty \subseteq R$. Suppose that every proper subset of $x, x_1, \ldots, x_k$ forms a regular sequence in S. Then*

$I^{[p]}S \subseteq (x_1^p, \dots, x_k^p)S + (x_1 \cdots x_k)^{p-1}IS$. Hence, by the equational forcing lemma, $I \subseteq (x_1, \dots, x_k)S$.

Proof. It is straightforward to check that $I^{[p]} \subseteq (x_1^p, \dots, x_k^p) :_R x^\infty$, and we apply the theorem on unraveling just above to write $(x_1^p, \dots, x_k^p) :_R x^\infty$ as a sum of several terms. In every term but the last there is a colon ideal involving a proper subset of the x's. These only become larger if computed in S, where these proper subsets are regular sequences. From this one sees that all the terms but the last are contained in $I^{[p]}S$. The last term is $(x_1 \cdots x_k)^{p-1}I$. Expanding to S yields the required result. $\square$

The proof of the unraveling theorem is given by an iteration of:

Lemma. *Let R be standard and let $x, x_1, \dots, x_h, y, z_{h+2}, \dots, z_k$ be a sequence of parameters such that*

(1) x is a colon-killer.

(2) The image of y is a colon-killer for R/I^{unmx}, where $I = (x_1, \dots, x_h)R$.

Let $I' = (z_{h+2}, \dots, z_k)R$. Then for every positive integer a,

$$(I + y^a R + I') :_R x^\infty = (I + I') :_R x^\infty + y^{a-1}\big((I + yR + I') :_R x^\infty\big).$$

We postpone the proof. Instead, we pause to give an application of the fact that R^+ is a big Cohen-Macaulay algebra in prime characteristic $p > 0$ for excellent local domains R. We need the following:

Proposition. *Let (R, m, K) be a regular local ring. Then an R-module B is flat over R if and only if every system of parameters for R is a (possibly improper) regular sequence on B.*

Proof. If suffices to show that $\mathrm{Tor}_j^R(M, B) = 0$ for every R-module M, and since M is direct limit of finitely generated R-modules, we may assume that M is finitely generated. This is certainly true for $j > n = \dim R$. Assume that the vanishing of Tor has been established for all $j > h \geq 1$, and that we want to show it for h as well. Now M has a finite filtration in which the factors are prime cyclic, and so it suffices to consider the case where $M = R/P$ with P prime. Choose a maximal regular sequence $x_1, \dots, x_s$ in P, where s is the height of P. Then P is a minimal prime of $(x_1, \dots, x_s)R$, and so R/P is an associated prime of $(x_1, \dots, x_s)R$. Thus, we have a short exact sequence $0 \to R/P \to R/(x_1, \dots, x_s) \to N \to 0$. The long exact sequence for Tor yields:

$$\mathrm{Tor}_{h+1}^R(N, B) \to Tor_h^R(R/P, B) \to \mathrm{Tor}_h^R(R/(x_1, \dots, x_s)R, B)$$

The first term vanishes by the choice of h, and the third term vanishes because the Koszul complex $\mathcal{K}_\bullet(x_1, \dots, x_s; R)$ is a free resolution of $R/(x_1, \dots, x_s)R$ and so

$$\mathrm{Tor}_h^R(R/(x_1, \dots, x_s)R, B) = H_h(\mathcal{K}_\bullet(x_1, \dots, x_s; R) \otimes B) = H_h(x_1, \dots, x_s;\, B) = 0,$$

since $x_1, \dots, x_s$ is a possibly improper regular sequence on B. $\square$

The vanishing theorem for maps of Tor. *Let $A \to R \to S$ be homomorphisms of Noetherian rings of prime characteristic $p > 0$ such that A and R are regular and R is a module-finite torsion-free extension of A. Then for every A-module M, the map $Tor_i^A(M, R) \to Tor_i^A(M, S)$ vanishes for $i \geq 1$.*

Proof. If not we can localize S at a prime in the support of a nonzero element of the image, A at the contraction P of that prime to A, and R at P as well. We can then replace A by $\widehat{A}$, R by $R \otimes_A \widehat{A}$, and S by $\widehat{S}$, and so assume that A, R, S are complete. Since R is torsion-free as a module over the domain A, all minimal primes have quotients of the same dimension as R. Then $R \to S$ factors $R \to R_1 \to S$ where R_1 is the quotient of R by a minimal prime contained in Ker $(R \to S)$. Since $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ factors $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, R_1) \to \mathrm{Tor}_i^A(M, S)$, it suffices to show that the right hand map is 0, and so we may replace R by R_1. Thus, we may also assume that R is a domain that is a module-finite extension of A. From this it follows that we may identify $A^+ \cong R^+$. Since A^+ is a big Cohen-Macaulay algebra over A, it is faithfully flat over A. Likewise, S^+ is faithfully flat over S. There is a commutative diagram:

$$\begin{array}{ccc} R^+ & \to & S^+ \\ \uparrow & & \uparrow \\ R & \to & S \end{array}$$

and so the compositions $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S) \to \mathrm{Tor}_i^A(M, S^+)$ and $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, R^+) \to \mathrm{Tor}_i^A(M, S^+)$ are the same. Since $R^+ \cong A^+$ is flat over A, we have that $\mathrm{Tor}_i^A(M, R^+) = 0$, and so both compositions must be 0. The result now follows because $\mathrm{Tor}_i^A(M, S) \to \mathrm{Tor}_i^A(M, S^+)$ is injective, which is a consequence of the fact that $S \to S^+$ is faithfully flat, since S^+ is a big Cohen-Macaulay algebra over the regular local ring S. $\square$