715: Lecture of September 25, 2002

First off, here is a lemma that we needed earlier (when proving the characteristic p version of the Ein-Lazarsfeld-Smith theorem on comparison of symbolic and ordinary powers of ideals) which I will fill in now.

Lemma. *Let M be a finitely generated module over a regular ring R. Then the set of associated primes of $F^e(M)$ is the same as the set of associated primes of M. In particular, the associated primes of $I \subseteq R$ are the same as those of $I^{[q]} \subseteq R$.*

Proof. If R/P embeds in M then, by the flatness of the Frobenius endomorphism, $R/P^{[q]}$ embeds in $F^e(M)$. Since P is the radical of $P^{[q]}$, R/P embeds in $R/P^{[q]}$ and, hence, in M.

For the other direction, note that M has a filtration by torsion-free modules over the various rings R/P as P runs through the associated primes of M. Each of these embeds in a free R/P-module. Thus, it suffices to show that if M is free over R/P, the only associated prime is $F^e(M)$ is P, and we need only consider the case $M = R/P$. But if $x \not in P$, we may apply F^e to the short exact sequence $0 \to R/P \to R/P$, where the map is multiplication by x, and we obtain that x^q is not a zerodivisor on $R/p^{[q]}$, and so x is not a zerodivisor on $R/P^{[q]}$. $\square$

Last time we showed that if I is m-primary, then $I^*R_m = (IR_m)^*$, and we also showed that if principal ideals of a ring R are tightly closed, then R is normal. (Reduced already follows from the fact that (0) is tightly closed.) We also noted that if $(R,\, m)$ is normal, then there is a decreasing sequence of m-primary irreducible ideals I_n cofinal with the power of m. (That is, R is *approximately Gorenstein.*) In the approximately Gorenstein case, the injective hull E of the residue field $K = R/m$ over R may be viewed as a direct limit of modules each of which is R/I_n for some integer n. In fact, if $(R,\, m,\, K)$ is local with E as the injective hull of the residue field, it is easy to show that $\mathrm{Hom}_R(R/I,\, E) \cong \mathrm{Ann}_E I$ is the injective hull of K over R/I. This means that $\mathrm{Ann}_E I_n$ may be identified with the injective hull of K over R/I_n. We can now quote the fact that an Artin local ring is injective as a module over itself if and only if it has a one-dimensional socle as a module over itself, in which case the ring itself is isomorphic to the injective hull of the residue field.

Lemma. *Let $(R,\, m,\, K)$ be local. Every finite length R-module M can be embedded in a finite direct sum of copies of E. Hence, if R is approximately Gorenstein, the every finite length R-module M can be embedded in a finite direct sum of copies of R/I_n for some n. The number of copies of E (respectively, of R/I_n) needed is the K-vector space dimension of the annihilator V of m in M (also called the socle of M).*

Proof. We have that $V \cong K^h$ for some h. Since K embeds in E, K^h embeds in E^h. Since E^h is injective, the map from $K^h \subseteq M$ extends to all of M. We claim that it is an injection: if $u \neq 0$ is in the kernel, then we can choose $t \geq 0$ as large as possible such that $m^t u \neq 0$ and then it follows that $m^t u$ is nonzero and killed by m, so that is has nonzero intersection with V. But then the map is not injective even on V, a contradiction.

Since each copy of E is a directed union of R/I_n, E^h will be the direct union of modules $(R/I_n)^h$. □

Corollary. *A local ring (R, m) is weakly F-regular if and only if it is normal and for some (equivalently, every) sequence of m-primary ideals I_n cofinal with the powers of m, $I_n = I_n^*$ for all n (or for all large n). Moreover, in this case, every submodule of every finitely generated R-module is tightly closed.*

Proof. It will suffice to show that if the ideals in some sequence as above are tightly closed for all $n \gg 0$, then every submodule of every finitely generated module M is tightly closed. Suppose, to the contrary, that $u \in N_M^*$. We may assume without loss of generality that $N = 0$ by passing to M/N. Then $u \neq 0$ and will fail to be in $m^t M$ for some large t, but it will still be in the tight closure of $m^t M$. Therefore, we get a new counterexample by passing to $M/m^t M$, and we may assume that M is finite length. But then $M \subseteq (R/I_n)^h$ for $n \gg 0$ and some h, and we get a counterexample in $(R/I_n)h$. The result now follows from the easy fact that if 0 is tightly closed in each of finitely many modules, then it is tightly closed in their direct sum. □

Corollary. *R is weakly F-regular if and only if for every maximal ideal m, every m-primary ideal is tightly closed. Also, R is weakly F-regular if and only if R_m is weakly F-regular for every local ring R_m of R at a maximal ideal. Finally, if R is weakly F-regular then every submodule of every module is tightly closed.*

Proof. Every ideal in a Noetherian ring is an intersection of m-primary ideals, from which the first statement follows. Second, the ideals primary to m are tightly closed if and only if their expansions to R_m are tightly closed in R_m, by a result proved last time.

Finally, suppose that u is in the tight closure of N in M. Replace N by a maximal submodule of M that does not contain u, and kill N. Then we have reduced to the case where u is in every nonzero submodule of M and is in the tight closure of 0. The hypothesis that there is an element u in every nonzero submodule of the Noetherian module M implies that the element is killed by a maximal ideal m of R, and the the module is killed by a power of m, and so may be thought of as a module over R_m. It is easy to see that the socle in this module is one-dimensional, generated by u. By an earlier lemma, this module embeds in R/I_n for some n. This means that M may be viewed as a submodule of R/I_n for a suitable m-primary ideal I_n. But then the fact that u is not in the tight closure of 0 in R/I_n implies that it is not in the tight closure of 0 in M, a contradiction. □

We also note the following:

Theorem. *Let R be a Noetherian ring, and let M be an R-module. Suppose one has a map $\alpha : R \to M$. (Then this map splits if and only if there is a map $\beta : M \to R$ such that $\beta \circ \alpha = id_R$, in which case M is the direct sum of $R \cong \alpha(R)$ and Ker β.) By the contraction of a submodule $N \subseteq M$ to R we mean $\alpha^{-1}(N)$.*

(a) *If (R, m, K) is complete local and K has injective hull E, then $R \to M$ splits if and only if the induced map $E \to M \otimes_R E$ is injective.*

(b) *If (R, m, K) is complete local and I_n is a sequence of m-primary ideals as in the definition of approximately Gorenstein, then $R \to M$ splits if and only if the contraction of I_nM to R is I_n for all n.*

(c) *If R is local and M is finitely generated, then $R \to M$ splits if and only if $\widehat{R} \to \widehat{M}$ splits.*

(d) *If R is local and M is finitely generated, then $R \to M$ splits if and only if $R_m \to M_m$ splits for every maximal ideal m of R.*

(e) *If M is finitely generated over R, and the local rings of R at maximal ideals are approximately Gorenstein (e.g., if R is normal) then $R \to M$ splits if and only if the contraction of IM to R is I for every irreducible m-primary ideal I of R (equivalently, for every ideal I of R).*

Proof. First note that $R \to M$ splits if and only if the induced map $\mathrm{Hom}_R(M, R) \to \mathrm{Hom}_R(R, R) = R$ is onto: in fact if and only if 1 is in the image. Part (c) follows because $\widehat{R}$ is faithfully flat over R, and tensoring with it commutes with Hom when M is finitely generated. Likewise, for (d), the surjectivity (i.e., whether the cokernel is zero) is local on the maximal ideals of R, and localization commutes with Hom when M is finitely generated.

For part (a), if $E \to M \otimes_R E$ is injective then we may Hom into the injective module E to conclude that $\mathrm{Hom}_R(M \otimes_R E, E) \to \mathrm{Hom}_R(E, E)$ is surjective. By the adjointness of tensor and Hom, the left term may be identified with $\mathrm{Hom}_R(M, \mathrm{Hom}_R(E, E)$ and $\mathrm{Hom}_R(E, E)$ may be identified with R when R is complete: this is the basic result of Matlis duality. Thus we get that $\mathrm{Hom}_R(M, \mathbb{R}) \to \mathrm{Hom}_R(R, R) = R$ is onto.

Part (b) follows because then E is the directed union of the R/I_n, and so if $R/I_n \to M \otimes_R R/I_n$ is injective for all n, we get that $E \to M \otimes_R E$ is injective.

Part (e) now follows from (d), (c), and (b) combined. $\square$