715: Lecture of September 26, 2002

The following result was asserted earlier, but we now have the material needed for a complete proof.

Corollary. *If R is weakly F-regular domain, then R is a direct summand of every module-finite extension domain.*

Proof. We know that R is normal, and so its local rings are approximately Gorenstein. Therefore, $R \subseteq S$ splits provided that $IS \cap R = I$ for all ideals I of R. But an earlier theorem asserts that $IS \cap R \subseteq I^*$ here, and $I^* = I$ since the ring is weakly F-regular. $\square$

Let (R, m, K) be a Cohen-Macaulay local ring of dimension d and let $x_1, \ldots, x_d$ be a system of parameters. Let $R_i = R/(x_1, \ldots, x_i)R$ (with $R_0 = R$) and note that x_{i+1} is not a zerodivisor on R_i for $0 \leq i < d$. The dimension of the socle in the Artinian factor ring $R_d = R/(x_1, \ldots, x_d)R$ is independent of the choice of the system of parameters and is called the *type* of R. (One way to see this is to show that it is the same as the vector space dimension of $\mathrm{Ext}^d_R(K, R)$, and that, in fact, the K-vector space dimension of $\mathrm{Ext}^{d-i}_R(K, R_i)$ is independent of the value of i. One can make the transition between consecutive terms by using the short exact sequence $0 \to R_i \to R_i \to R_{i+1} \to 0$, where the map on R_i is multiplication by x_{i+1}, and then using the long exact sequence for Ext.)

We define a Cohen-Macaulay local ring to be *Gorenstein* if it has type one, which means that if we kill the ideal generated by some (equivalently, any) system of parameters, the resulting Artinian ring has a socle of K-vector space dimension one.

Note that Gorenstein rings are clearly approximately Gorenstein, for if $x_1, \ldots, x_d$ is a system of parameters, one may take $I_n = x_1^n, \ldots, x_d^n$.

It is an open question whether for "nice" (e.g., locally excellent) domains R of characteristic $p > 0$, one has that R is weakly F-regular if and only if R is a direct summand of every module-finite extension domain. This is known to be true in the case of a Gorenstein ring.

While we have shown already that test elements exist for module-finite extension domains of regular domains when the extension of fraction fields is separable, we want to show next that there are many other "sources" of test elements. One way to develop these is through the theory of strongly F-regular rings. One consequence of the theory is that if R is essentially of finite type (i.e., a localization of a finite type algebra) over an excellent local ring and is reduced, then every element $c \in R^\circ$ such that R_c is regular has a power that is a test element. In fact, we shall prove much more. This theory is only defined for F-finite rings, and so we first discuss this notion.

A Noetherian ring R of positive prime characteristic p is called *F-finite* if R is a module-finite extension of its subring R^p. We leave the following as exercises:

A homomorphic image or localization of an F-finite ring is F-finite.

A polynomial or power series ring in one or finitely many variables over an F-finite ring is F-finite. (The case of one variable implies the case of finitely many.)

A complete local ring R is F-finite if and only if its residue field K is F-finite (if K is F-finite, R will be a homomorphic image of a formal power series ring in finitely many variables over K).

Later, we shall show that any complete local ring has a faithfully flat purely inseparable extension that is F-finite over the original ring and that this extension can be chosen so that it shares many of the good properties of the original ring. Thus, the case of F-finite rings is not as restrictive as it might appear at first.

We say that an F-finite Noetherian domain R of positive characteristic p is *strongly F-regular* if for every nonzero element $c \in R$ there exists a power q_0 of p such that the map $R \to R^{1/q_0}$ that sends 1 to c^{1/q_0} splits over R, i.e., there is an R-linear map $\phi : R^{1/q_0} \to R$ that sends c^{1/q_0} to 1. We note the following facts:

(1) Once we have that there is a splitting for one such q_0, we get that the inclusion map $R \subseteq R^{1/q}$ splits for every q, and that there is a splitting of the map $R \to R^{1/q}$ that sends 1 to $c^{1/q}$ for all $q \geq q_0$.

(2) If R is strongly F-regular, then R is weakly F-regular.

(3) If R is strongly F-regular, then so is every localization of R. Thus, strongly F-regular rings are F-regular.

(4) If R is an F-finite Gorenstein domain, then R is strongly F-regular if and only if R is weakly F-regular.

Before proving these statements, we want to note that it is conjectured that weakly F-regular F-finite domains are strongly F-regular in general, not just in the Gorenstein case. If this conjecture is true, it implies that weakly F-regular and F-regular are equivalent for locally excellent rings: we shall discuss this some more later. Considering that strongly F-regular rings will also be a source of test elements, these facts provide plenty of motivation for studying them.

We next prove (1) above. Given ϕ as in (1) we get an R-linear map $R^{1/q_0} \to R$ sending 1 to 1 by mapping s to $\phi(c^{1/q_0}s)$. We may restrict this to $R^{1/p}$ to get a splitting of $R \subseteq R^{1/p}$ over R. Taking qth roots then gives a splitting of $R^{1/q} \subseteq R^{1/pq}$ over $R^{1/q}$ for all q, and hence over R. By composing several of these splittings we see that $R \to R^{1/q}$ splits for all q. Moreover, given ϕ, we may take pth roots to get a splitting of the map $R^{1/p} \to R^{1/pq_0}$ (over $R^{1/p}$) that sends 1 to c^{1/pq_0}. Since the map $R \subseteq R^{1/p}$ splits over R, we may compose the two splittings to get an R-linear map $R^{1/pq_0} \to R$ that sends c^{1/pq_0} to 1. That one gets such a splitting for all $q \geq q_0$ now follows by induction. $\square$