715: Lecture of September 30, 2002

Before continuing with the proofs of the assertions from last time about strong F-regularity, we want to give two simple applications of the existence of test elements that show how important it is to know that they exist for theoretical reasons rather than reasons connected with calculation.

Proposition. *Suppose that R has a test element c and that $d \in R^\circ$. Suppose that $du^q \in I^{[q]}$ (or even in $(I^{[q]})^*$) for infinitely many values of q. Then $u \in I^*$.*

Proof. First observe that if $du^Q \in I^{[Q]}$ for a certain $Q = p^E$, then for any smaller q we can write $Q = qq'$, and then $du^q \in (I^{[q]})^F$ (the Frobenius closure), since $(du^q)^{q'} = d^{q'-1}(du^Q) \in I^{[Q]} = (I^{[q]})^{[q']}$, from which it follows that $du^q \in (I^{[q]})^F$. But $(I^{[q]})^F \subseteq (I^q)^*$, and so we have that $du^q \in (I^{[q]})^*$ for all q. But then $cdu^q \in I^{[q]}$ for all q, from which it follows that $u \in I^*$.

Similarly, if $du^q \in (I^{[q]})^*$ for infinitely many q, then $cdu^q \in I^{[q]}$ for infinitely many q, and the result follows from what we have already proved. $\square$

Proposition. *Suppose that R is a local domain for which we know*[1] *that if $x_1, \ldots, x_{k+1}$ is part of a system of parameters, then then $(x_1, \ldots, x_k) : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$. Suppose also that R has a test element. Then $(x_1, \ldots, x_k)^* : x_{k+1} \subseteq (x_1, \ldots, x_k)^*$.*

Proof. Suppose that $x_{k+1}u \in (x_1, \ldots, x_k)^*$. Then for $d \neq 0$ and all $q \gg 0$, we have that $dx_{k+1}^q u^q \in (x_1^q, \ldots, x_k^q)$, and so $du^q \in (x_1^q, \ldots, x_k^q) : x_{k+1}^q \subseteq (x_1^q, \ldots, x_k^q)^*$, since $x_1^q, \ldots, x_{k+1}^q$ is also part of a system of parameters. But then, by the parenthetical part of the preceding Proposition (or by multiplying by a test element c), we get that $u \in (x_1, \ldots, x_k)^*$. $\square$

We now continue with the proofs of the assertions of the previous lecture.

We want to show that if R is strongly F-regular then R is weakly F-regular. Suppose that $cu^q \in I^{[q]}$ for all $q \gg 0$, where $c \neq 0$. Then, taking qth roots, $uc^{1/q} \in IR^{1^{[q]}}$ for all $q \gg 0$. But we also now that for all $q \gg 0$, there is an R-linear map $R^{1/q} \to R$ sending $c^{1/q}$ to 1. Applying such a map ϕ for any sufficiently large q shows that $u = u\phi(c^1/q) \in I$, so that $u \in I^* \Rightarrow u \in I$, as required. $\square$

We further want to show that if R is strongly F-regular, then so is every localization $W^{-1}R$ of R. First note that $W^{-1}(R^{1/q}) \cong (W^{-1}R)^{1/q}$: the only problem is that the latter might be bigger, and it suffices to see that qth roots of elements of the form $1/w$ with $w \in W$ are in the former. But if $v^q = w$, the qth root of $1/w$ is $1/v = v^{q-1}/v^q = v^{q-1}/w \in$

[1]For example, we know this if R is module-finite over a regular local ring, and we shall prove it later for every excellent local domain.

$W^{-1}(R^{1/q})$, as needed. Now consider a nonzero element $c/w \in W^{-1}R$, where $c \in R - \{0\}$ and $w \in W$. Then for some q there is an R-linear map $\phi : R^{1/q} \to R$ and sending $c^{1/q}$ to 1. The map ϕ induces a $W^{-1}R$-linear map $W^{-1}R^{1/q} \to R$ sending $c^{1/q}$ to 1, and we denote this map by ϕ as well. Since $W^{-1}R^{1/q} \cong (W^{-1}R)^{1/q}$, and that the map sending $s \in W^{-1}R^{1/q}$ to $\phi(w^{1/q}s)$ has the required property. $\square$

In order to study the Gorenstein case, we first want to prove a result that permits us to reduce to the local case.

Lemma. *Let R be an F-finite Noetherian domain of characteristic $p > 0$. Then R is strongly F-regular if and only if R_m is strongly F-regular for every maximal ideal m of R.*

Proof. Since we have already shown that all localizations of strongly F-regular rings are F-regular, we only need to prove the "if" part. Let $c \in R - \{0\}$ be given. For every maximal ideal m of R there exists an power, $q(m)$ of p, and an R-linear map $(R_m)^(1/q(m)) \cong (R - m)^{-1}R^{1/q(m)} \to R_m$ that sends $c^{1/q(m)} \to 1$. We may restrict the map to $R^{1/q(m)}$: since this is a finitely generated R-module, the image is contained in R_a for a single element $a \in R - m$. It follows that the map $R \to R^{1/q(m)}$ sending $1 \to c^{1/q(m)}$ splits on a Zariski open neighborhood of m. Since the maximal spectrum of R is quasi-compact, we can choose a finite cover U_i by open sets such that on each open set U_i, the map $R \to R^{1/q_i}$ sending 1 to c^{1/q_i} splits for a certain power q_i of p. If q denotes the maximum of the finitely many q_i, we have that the map $R \to R^{1/q}$ sending 1 to $c^{1/q}$ splits locally for every maximal ideal m of R. This implies that it splits. $\square$

We can now prove:

Theorem. *Let R be an F-finite Gorenstein ring (i.e., the local rings of R are Gorenstein). Then R is weakly F-regular if and only if it is strongly F-regular if and only if it is F-regular. If R is local of Krull dimension d and $x_1, \dots, x_d$ is a system of parameters, then the following conditions are equivalent:*
(a) *The ideal $I = (x_1, \dots, x_d)R$ is tightly closed in R.*
(b) *R is strongly F-regular.*
(c) *R is F-regular.*
(d) *R is weakly F-regular.*

This will give a very useful check for F-regularity in Gorenstein local rings: it suffices to check whether the ideal generated by a single system of paramters is tightly closed.

We first note the following fact, which is left as an exercise: if $x_1, \dots, x_d$ is a regular sequence on an R-module M, and $I_t = (x_1^t, \dots, x_d^t)R$, then the map given by multiplications by $x = x_1 \cdots x_d$, the product of the elements in the regular sequence, is an injection of M/I_tM into $M/I_{t+1}M$ for all t. Hence, multiplication by x^k injects M/I_tM into $M/i_{t+k}M$ for all integers $k, \geq 1$, and, in particular, injects M/I_1M into $M/I_{k+1}M$ for every integer $k \geq 1$. When R is a Gorenstein local ring and $x_1, \dots, x_d$ is a system of parameters, it follows that multiplication by $(x_1 \cdots x_d)^{t-1}$ injects $R/(x_1, \dots, x_d)R$ into R/I_tR. Therefore, if $u \in R$ generates the socle in $R/(x_1, \dots, x_d)R$, we must have that $(x_1 \cdots x_d)^{t-1}u$ generates the socle in R/I_t.

Proof of the Theorem. We first consider the local case, where the implications (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are trivial. It suffices to prove that (a) $\Rightarrow$ (b). Assume (a), and suppose, to the contrary, that $c \in R - \{0\}$ is such that the map $R \to R^{1/q}$ sending 1 to $c^{1/q}$ fails to split for all q. Then for all q there is a positive integer t such that $I_t R^{1/q}$ contracted to $c^{1/q}R$ is larger than $c^{1/q}I_t$. Suppose that $u \in R$ represents a socle generator in R/I_1. Then $(x_1 \cdots x_d)^{t-1}u$ represents a socle generator in R/I_t. The failure of contraction for this q and t says that $x^{t-1}uc^{1/q} \in I_t R^{1/q}$. But since the x_j are a regular sequence on $R^{1/q}$, from this we get that $uc^{1/q} \in I_1 R^{1/q}$ for all q. Even though t depended on q in the preceding statement, this second statement only involves q. Taking q th powers, we have that $cu^q \in I_1^{[q]}$ for all q. But then $u \in I_1^* = I_1$, a contradiction. This takes care of the local case.

Since we already know that strong F-regularity passes to localizations of R and may be tested locally in local rings at maximal ideals, and we also know that weak F-regularity may be tested locally at maximal ideals, the general case follows easily. $\square$