In all of these problems assume that all given rings are Noetherian of characteristic p, unless otherwise specified. R always denotes a given ring, and I an ideal of R.

1. Let M be a Noetherian module over a Noetherian ring (characteristic p is not needed), and suppose that $u \in M - \{0\}$ is in every nonzero submodule of M. Prove that u is killed by a maximal ideal m of R, and that M is killed by a power of m.

2. Prove the statement made in class that the following two assertions about $u \in R$, $I \subseteq R$ are equivalent (keep in mind that R has only finitely many minimal primes). [Note: this shows that our two definitions of tight closure are equivalent. The extension to the case of modules is easy.]

(1) There is an element c not in any minimal prime of R such that $cu^q \in I^{[q]}$ for all $q \gg 0$.

(2) For every minimal prime P of R, if $S = R/P$ then the image of u is in the tight closure of $I(R/P)$ in R/P, i.e., there exists $c_P \in R - P$ such that $c_P u^q \in I^{[q]} + P$ for all $q \gg 0$.

3. Let $R = K[x_1, \ldots, x_n]/(g)$ where $g = \sum_{j=1}^n x_j^{d_j}$. Prove or disprove: $x_n^{d_n - 1}$ is in the tight closure of $I = (x_1, \ldots, x_{n-1})$ if and only if $\displaystyle\sum_{j=1}^n \frac{1}{d_j} \leq 1$.

4. Prove that localizations and homomorphic images of F-finite rings are F-finite, and that adjunction of one (or finitely many) polynomial or formal power series variables also preserves this property.

5. Let $x_1, \ldots, x_k$ be part of a system of parameters for a local domain, and suppose we know (as we do in many cases) that $(x_1, \ldots, x_j)^* : x_{j+1} \subseteq (x_1, \ldots, x_j)^*$, $0 \leq j < k$. Show that if $x_1, \ldots, x_k$ is tightly closed, then so is each initial segement $(x_1, \ldots, x_j)$, $0 \leq j \leq q$, and that $x_1, \ldots, x_k$ is a regular sequence. (The key point is to show that if $(x_1, \ldots, x_k)$ is tightly closed, then so is $(x_1, \ldots, x_{k-1})$.)