1. Consider any associated prime P of M. Then R/P embeds in M. If $a \in R/P - \{0\}$ is not a unit, it is easy to check that $\bigcap_n a^n(R/P) = 0$, contradicting the fact that u should be in all of these. Thus, R/P must be a field, so that every associated prime of M must be a maximal ideal. If there are two associated primes, m_1 and m_2, both maximal, then u must be in both of $\mathrm{Ann}_M\, m_1$ and $\mathrm{Ann}_M\, m_2$, and so killed by $m_1 + m_2 = R$, a contradiction. Thus, M has a unique associated prime m, and it is maximal, which shows that M has finite length and is killed by a power of m. Since $u \in \mathrm{Ann}_M m$, u is killed by m.

2. It is clear that (1) $\Rightarrow$ (2), since we may take each c_P to be the image of c. Now suppose that we have c_P as in (2) for each minimal prime p. Choose an element d_P in R such that $d_P \equiv c_P \bmod P$ and $d_P \in R^{\mathrm{o}}$. (If we could not do this, then $c_P + P \subseteq \bigcup_j P_j$, the union of the minimal primes, and by a standard lemma on prime avoidance, this would imply that $c_P + P$ is contained in one of them, a contradiction.) Choose an element in each of the minimal primes P' other than P that is not in P, and multiply them together to get, for each minimal prime P, an element b_P that is in every minimal prime but P. Then $c_P b_P$ is in every minimal prime but P, and so $c_P b_P u^q \in I^{[q]} + b_P P$ for all P. Since $b_p P$ is contained in every minimal prime, it is contained in the ideal of nilpotents N. Adding, we find that with $c = \sum_P c_P b_P$, we have that $c \in R^{\mathrm{o}}$ and $cu^q \in I^{[q]} + N$ for all $q > q_0$, say. There will be some power of p, call it q', such that $N^{[q']} = 0$, and then $c^{q'} u^{qq'} \in I^{[qq']}$ for all $q \geq q_0$, which says that $c^{q'} u^Q \in I^{[Q]}$ for all $Q \geq q_0 q'$, as required.

3. Let $x_n = x$ and $d_n = d$. $x^{d-1} \in I^*$ is true if $\delta = \sum_{j=1}^n \frac{1}{d_j} \leq 1$. If $\delta > 1$ and the char. $p \gg 0$ (but think of all the d_j as fixed), then $x^{d-1} \notin I^*$: but to answer the problem it suffices to give one example where $\delta > 1$ and $x^{d-1} \notin I^*$, e.g., $k[x_1, x_2, x_3]/(x_1^2 + x_2^2 + x_3^2)$ when K has char. 2. Then $x_1 + x_2 + x_3$ is nilpotent and so in the tight closure of every ideal, which shows that the tight closure of (x_1, x_2) will contain x_3.

Now suppose that $\delta \leq 1$ (with $n \geq 2$, all $d_j \geq 2$) and let $q = ad + r$ where $a > 0$ is an integer and $0 \leq r < d$. Take $c = x^r$. Then $x^r(x^{d-1})^q = x^{dq-ad-r+r} = (x^d)^{q-a} = (x_1^{d_1} + \cdots + x_{n-1}^{d_{n-1}})^(q-a)$, and the expansion is the sum of terms $\binom{q-a}{b_1,\,\dots,\,b_{n-1}} x_1^{d_1 b_1} \cdots x_{n-1}^{d_{n-1} b_{n-1}}$, where the b_j are nonnegative integers whose sum is $q - a$. If every $d_j b_j \leq q-1$, then every $b_j \leq (q-1)/d_j$, which shows implies that $q - a \leq \sum_{j=1}^{n-1} (q-1)/d_j \leq (q-1)(1-1/d)$, or $q - (q-r)/d \leq (q-1)(1-1/d)$, i.e., $dq - q + r \leq (q-1)(d-1)$, or $(d-1)q + r \leq (q-1)(d-1)$, or $d - 1 + r \leq 0$, a contradiction, since $d \geq 2$.

4. If $u_1, \dots, u_n$ span R over R^p this remains true for $W^{-1}R$ over $W^{-1}(R^p) \cong (W^{-1}R)^p$, and for R/J over $(R/J)^p$. Moreover, the products of these together with the elements x^i, $0 \leq i < p$ span $R[x]$ over $(R[x])^p = R^p[x^p]$, and also span $R[[x]]$ over $R[[x]]^p \cong R^p[[x^p]]$. The case of several variables follows at once by induction from the case of one.

5. Let $J = (x_1, \dots, x_{k-1})$, and let $x = x_k$. Then $J^* \subseteq (J + xR)^*$ which is given to be tightly closed, so that $J^* \subseteq J + xR$. But if $u \in J^*$ and $u = j + xr$, with $j \in J$ and $r \in R$, then $xr = u - j \in J^*$, and so $r \in J^* :_R xR$, which we know from our hypotheses to be the same as J^*. Thus, $J^* \subseteq J + xJ^*$, and since $\supseteq$ is obvious, we have $J^* = J + xJ^*$. By Nakayama's lemma applied to J^*/J, we have that $J^*/J = 0$, i.e., $J^* = J$, as required. Working backwards, we find by induction that every initial segment $(x_1, \dots, x_j)$ is tightly closed. But then $(x_1, \dots, x_j) :_R x_{j+1} \subseteq (x_1, \dots, x_j)^* = (x_1, \dots, x_j)$, and so the sequence is regular.