Math 715, Fall 2002 Problems: #2 Prof. Hochster

In all of these problems assume that all given rings are Noetherian of characteristic p, unless otherwise specified. R always denotes a given ring, and I an ideal of R. As usual, $q = p^e$ with $e \in \mathbb{N}$.

1. Show that an element of R° which is a test element for every local ring of R is a test element for every localization of R, including R itself.

2. Let S be faithfully flat over R. Show that if $c \in R$ is a test element for S, then c is a test element for R. Hence, if R is local, a test element for $\widehat{R}$ is a test element for R. Conclude that a completely stable test element is a test element.

3. Let R be a local F-finite domain. Verify that the map $A = \widehat{R} \otimes_R R^{1/q} \to \widehat{R}^{1/q} = B$ is an isomorphism. [We are dealing with excellent rings, so that completions of reduced rings are reduced. The ring A is the same as the completion of $R^{1/q}$, since $R^{1/q}$ is module-finite over R. Injectivity: use that if an element is in the kernel, its pth power is in the kernel. Surjectivity: any element b of B is the qth root of the limit of a Cauchy sequence of elements of R; qth roots of the elements in the Cauchy sequence exist in A (check that those qth roots form a Cauchy sequence in A whose limit will map to b).]

4. Let $T = k[x, y, u, v]$ be a polynomial ring in four variables over a field k, and let $S = T_m$, where $m = (x, y, u, v)$. Let $R = T/(xv - yu)$. Determine the Hilbert-Kunz function $\mathrm{HK}_m(e)$ for the R-module R.

5. S be a Noetherian ring of positive characteristic p that has a completely stable test element c, and let $N \subseteq M$ be finitely generated R-modules. Let $u \in M$.

(a) Show that $u \in N^*_M$ if and only if whenever T is the completion of a local ring of R, $1 \otimes u$ is in the tight closure over T of the image of $T \otimes_S N$ in $T \otimes_S M$.

(b) Let $R \to S$ be a homomorphism of Noetherian rings of positive prime characteristic. Show that if S has a completely stable test element then persistence of tight closure holds from R to S.