1. Let c be the element: we first show that it is a test element for R. Let I be an ideal of R and suppose that $u \in I^*$. Then $u/1 \in (IR_P)^*$ for all P by a class result. If $cu^q \notin I^{[q]}$, we can preserve this by localizing at a prime in the support of $(I^{[q]} + cu^q R)/I^{[q]}$, and this will contradict that c is a test element in R_P. This proves the result for R itself. But the local rings of any $S^{-1}R$ are a subset of the local rings of R. $\square$

2. We identify R with its image in S: faithfully flat homomorphisms are injective. If $u \in I^*$ in R, then $u \in (IS)^*$ by an observation in class: R^{o} maps into S^{o}. Thus, for all q, $cu^q \in (IS)^{[q]} \cap R = I^{[q]}S \cap R = I^{[q]}$ (since ideals are contracted from faithfully flat extensions), as required. The second statement follows because $R \to \widehat{R}$ is faithfully flat, and the third statement is immediate from the second and **1.**

3. Injectivity is immediate: the qth power of any any element of A is in $\widehat{R}$. If the element is in the kernel and is nonzero, so is the qth power, since A is reduced. But $\widehat{R} \to \widehat{R}^{1/q}$ is obviously injective. For surjectivity, suppose $b \in \widehat{R}$, and write b as the limit of a Cauchy sequence $\{r_n\}_n$ in R. Consider the sequence $\{r_n^{1/q}\}_n$ in $R^{1/q}$. If $r_n - r_m \in m^N$, then $r_n^{1/q} - r_m^{1/q} \in m_0^N$, where m_0 is the maximal ideal of $R^{1/q}$, and is the same as the set $\{u^{1/q} : u \in m\}$. Then $\{r_n^{1/q}\}_n$ is clearly a Cauchy sequence in $R^{1/q}$, since $\{r_n\}_n$ is in R, and converges to some element α. Then $\{r_n\}_n$ converges to α^q which means that $\alpha^q = b$, and so $\alpha = b^{1/q}$, as required.

4. A K-basis for the ring consists of those monomials such that y, u do not both occur. Killing the ideal generated by the qth powers of x, y, u and v leaves at most those monomials of this form such that all exponents are less than q, but there is the additional constraint that if $x^a v^b y^c$ occurs then $\min\{a, b\} + c < q$, and the same constraint applies for $x^a v^b u^c$. If $a = b$, there are $q - a$ choices for c, namely $q - 1 - a, \ldots, 0$ which give $\sum_{a=0}^{q-1}(q-a) = \sum_{j=1}^{q} = \binom{q+1}{2}$ possibilities. If $a < b$ (think of a as fixed) there are $q-1-a$ choices for b, namely $a+1, \ldots, q-1$, and $q - a$ choices for c, as before, which lead to $\sum_{a=0}^{q-1}(q-a-1)(q-a) = \sum_{j=1}^{q} 2\binom{j}{2} = 2\binom{q+1}{3}$ possibilities. Reversing the roles of a and b doubles this number. We get $(q+1)q/2 + 4(q+1)q(q-1)/6$ monomials such that y occurs but not u, and the same number such that u occurs but not y, but we cannot simply double because we would be counting the number of monomials such that neither y nor u occurs twice: this number is q^2. Thus, the total is

$$(q+1)q + 4(q+1)q(q-1)/3 - q^2 = (q/3)\big((3q+3) + (4q^2-4) - 3q\big) = q(4q^2-1)/3,$$

where $q = 2^e$. The Hilbert-Kunz multiplicity is $4/3$.

5. (a) The "only if" part is clear, since the completed local rings are flat, and persistence is automatic for flat base change. On the other hand, if u is in the tight closure for each completed local ring, then $cu^q \in N^{[q]}$ for all q: if not, one can preserve this failure for some localization, and then it will be preserved as well when one completes, which contradicts the fact that c is a test element in the completed localization.

(b) If tight closure fails over S it must also fail when we map further to a completed local ring of S, by part (a). Therefore, there is no loss of generality in assuming that S is a complete local ring. We may further kill a minimal prime of S and get failure of persistence when S is a complete local domain. Let P be the contraction of the maximal ideal of S to R. Then tight closure persists when we pass from $R \to R_P \to \widehat{R_P}$, since these are flat, and so it must fail for the map $\widehat{R_P} \to S$. We may also replace $\widehat{R_P}$ by its quotient by a minimal prime contained in the kernel. But persistence holds for maps from a complete local domain, by the theorem on persistence of tight closure in the presence of many test elements from the notes on the lecture of October 28, and the fact that in every reduced ring T essentially of finite type over an excellent local ring, if $c \in T^{\circ}$ is such that T_c is regular, then c has a power that is a completely stable test element in T.