In all of these problems assume that all given rings are Noetherian of characteristic p, unless otherwise specified. R always denotes a given ring, and I an ideal of R. As usual, $q = p^e$ with $e \in \mathbb{N}$.

1. Let $R \subseteq S$ be a map of domains such that $IS \cap R = I$ for every ideal I of R. Show that:

(a) If S is weakly F-regular then R is weakly F-regular.

(b) If $c \in R$ is a test element in S then it is a test element in R.

2. Let R and S be finitely generated algebras over an algebraically closed field K such that R and S are reduced.

(a) Show that $T = R \otimes_K S$ has a test element of the form $c \otimes d$, where c is a test element for R and d is a test element for S. (Note: $(R \otimes_K S)_{c \otimes d} \cong (R_c) \otimes_K (S_c)$. Also observe that if R_c and S_d are both regular, then $R_c \otimes_K S_d$ is also regular (explain — note that this is not necessarily true when K is not algebraically closed, e.g., if R and S are both purely inseparable field extensions of K).

(b) Prove that if R and S are both weakly F-regular then so is T.

3. Let R be a finitely generated algebra over a field K such that R is $\mathbb{N}$-graded, with $R_0 = K$. Assume that K is infinite.[1] Prove that the test ideal in R is homogeneous. (For every element $\alpha \in K - \{0\}$, R has an automorphism that sends $\sum_{j=0}^{k} f_j$, where every f_j is homogeneous of degree j, to $\sum_{j=0}^{k} \alpha^j f_j$. Show that any ideal that is mapped to itself by all automorphisms of R is homogeneous.)

4. Let I be an ideal of R that is generated by monomials in the elements $x_1, \ldots, x_n$ of a *permutable* regular sequence in R. Prove that for any multiplicative system S in R, $(IS^{-1}R)^* = (I^*)S^{-1}R$. (It may be helpful to write R as an intersection of larger ideals, each of which is generated by a regular sequence.)

5. Let (R, m, K) be an excellent reduced Gorenstein local ring of Krull dimension d, and suppose that R_P is weakly F-regular for every prime P strictly contained in m. (This is true, in particular, if R has an isolated singularity.)

(a) Prove that R has a system of parameters $f_1, \ldots, f_d$ such that every f_j is a test element.

(b) Choose a system of parameters as in (a) consisting of test elements, and let I be the ideal they generate. Prove that $I :_R I^*$ is the test ideal of R.

[1]The result is true, but harder to prove, when K is finite.