1. (a) If I is any ideal of R then $cu^q \in I^{[q]}$ for all $q \gg 1$ implies that $cu^q \in (IS)^{[q]}$ for all $q \gg 1$, and so $u \in (IS)^* \cap R = IS \cap R = I$. $\square$

(b) If $u \in I^*$ in R, then $u \in (IS)^*$ and so for all q, $cu^q \in I^{[q]}S \cap R = I^{[q]}$. $\square$

2. (a) Any maximal ideal Q of $R \otimes_K S$ has the form $m \otimes_K S + R \otimes_K n$ where m, n are maximal in R and S respectively. Then $R_m \to (R \otimes_K S)_Q$ is (faithfully) flat local, with closed fiber $K \otimes_K S_n$. This shows that if R and S are both regular, so is T. (When K is not algebraically closed one may be tensoring with an inseparable field extension of K, so that regularity may be lost.) Choose c' in R^{o} and $d' \in S^{\mathrm{o}}$ such that $R_{c'}$ and $S_{d'}$ are regular. Then $R_{c'} \otimes_K S_{d'} \cong T_{c' \otimes d'}$ is regular, and it follows that $c' \otimes d'$ has a power $c \otimes d$ that is a test element: taking the power still larger if necessary, c will be a test element in R and d a test element in S. $\square$

(b) If I is irreducible in R and inside m^N and J is irreducible in S inside n^N, then $\mathfrak{A} = I \otimes_K S + R \otimes_K J$ is irreducible T and inside $(m \otimes_K S + R \otimes_K n)^N$. (Irreducibility follows because the quotient by $\mathfrak{A}$ has the form $(R/I) \otimes_K (S/J)$, which is flat local over R/I (type 1) with closed fiber S/J (type 1). The type of $(R/I) \otimes_K (S/J)$ is therefore $1 \cdot 1 = 1$.) It suffices to prove that an ideal of this form is tightly closed, and it is easy to see that if the image of $u \in R$ generates the socle in R/I and the image of $v \in S$ generates the socle in S/J, then the image of $u \otimes v$ in T generates the socle mod $\mathfrak{A}$. Thus, we need only see that, for a test element $c \otimes d$ as in part (a), $(c \otimes d)(u \otimes v)^q \notin \mathfrak{A}^{[q]}$ for $q \gg 1$, i.e., that $cu^q \otimes dv^q \notin I^{[q]} \otimes S + R \otimes_K J^{[q]}$ for large q. This follows because for large q, $cu^q \notin I^{[q]}$ and $du^q \notin J^{[q]}$. $\square$

3. Evidently, the test ideal τ is mapped to itself by every ring automorphism of R. Any such ideal τ in an $\mathbb{N}$-graded ring R in which R_0 contains an infinite field is homogeneous — one sees this as follows. For every nonzero element a of the field there is a K-automorphism θ_a of the ring that sends f_d to $a^d f_d$ for each d-form f. If $f_0 + \cdots + f_h$ is in the ideal, then we may choose distinct nonzero elements $a_0, \ldots, a_h$ of K, and it follows that all of the elements $\sum_{j=0}^{h} a_i^j f_j = u_i \in \tau$. The invertibility of the size $h+1$ square matrix a_i^j as i, j vary in $\{0, \ldots, h\}$ enables one to solve for the f_j as K-linear combinations of the u_i, and so they are in τ. (A non-trivial linear relation on the rows of the matrix yields a non-trivial polynomial of degree h with at least $h+1$ roots, a contradiction.)

4. By the argument in the paragraph beginning near the bottom of p. 3 in the Lecture Notes for October 10, most of which is on p. 4, an ideal $\mathfrak{A}$ generated by monomials in a permutable regular sequence is an intersection $\mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_h$ of several ideals each of which is generated by powers of terms in the regular sequence. We next note that for ideals generated by monomials in a permutable regular sequence, finite intersection commutes with tight closure. The non-trivial inclusion $\mathfrak{B}_1^* \cap \cdots \cap \mathfrak{B}_h^* \subseteq (\mathfrak{B}_1 \cap \cdots \cap \mathfrak{B}_h)^*$ follows because if $c_i u^q \in \mathfrak{B}_i^{[q]}$ for all $q \gg 1$, then $(c_1 \cdots c_h)u^q \in \bigcap_i \mathfrak{B}_i^{[q]}$, which is easily verified, since operations on these monomial ideals are "formal" as described in the Lecture Notes from October 10, to be $(\bigcap_i \mathfrak{B}_i)^{[q]}$. But then $S^{-1}(\mathfrak{A}^*) = S^{-1}(\bigcap_i \mathfrak{B}_i)^* = S^{-1}(\bigcap_i \mathfrak{B}_i^*) = \bigcap_i \bigl(S^{-1}(\mathfrak{B}_i^*)\bigr) = \bigcap_i (S^{-1}\mathfrak{B}_i)^*$ (by the Theorem in the Lecture Notes for November 14) $= (\bigcap_i S^{-1}\mathfrak{B}_i)^* = \bigl(S^{-1}(\bigcap_i \mathfrak{B}_i)\bigr)^* = (S^{-1}\mathfrak{A})^*$. Note that the regular sequence is possibly improper after localizing after S, but it is a possibly improper regular sequence no matter how we order the terms, and the results of the Lecture of October 10 on computing "formally" are valid.

5. (a) By the final Theorem of the Lecture Notes for November 14, every element of m that is in R^{o} has a power that is a test element. Thus, if we choose any system of parameters, each element has a power that is a test element.

(b) Let $I = (x_1, \ldots, x_d)$. Clearly, any element in the test ideal multiplies I^* into I. Now let $u \in I :_R I^*$. It will suffice to show that u multiplies J^* into J for every ideal J of R. Suppose not: let $y \in J^*$ be such that $uy \notin J$. Choose $J_1 \supseteq J$ maximal in R such that $uy \notin J_1$. Then we may replace J by J_1. Now J is an m-primary irreducible ideal, and so R/J embeds in the injective hull of $K = R/m$ over R, and we may view this injective hull as the directed union of cyclic modules of the form $R/(x_1^t, \ldots, x_d^t)$, Let $z \in R$ be such that the image of y in R/J maps to the class of z in $R/(x_1^t, \ldots, x_d^t)$ in a suitable injection of $R/I \hookrightarrow E$ that factors $R/J \hookrightarrow R/(x_1^t, \ldots, x_d^t) \hookrightarrow E$. Then $z \in (x_1^t, \ldots, x_d^t)^*$, but $uz \notin (x_1^t, \ldots, x_d^t)$. Since $x_1, \ldots, x_d$ are all test elements, each $x_i z \in (x_1^t, \ldots, x_d^t)$, from which we conclude that $z \in (x_1^t, \ldots, x_d^t)R :_R (x_1, \ldots, x_d)R = (x_1^t, \ldots, x_d^t)R + (x_1 \cdots x_d)^{t-1}R$, since we may compute the colon ideal formally. If we write $z = w + (x_1 \cdots x_d)^{t-1}r$ with $w \in (x_1^t, \ldots, x_d^t)R$, we see that $(x_1 \cdots x_d)^{t-1}r \in (x_1^t, \ldots, x_d^t)^*$, and so we find that for some $c \in R^{\mathrm{o}}$, $c(x_1 \cdots x_d)^{qt-q}r^q \in (x_1^{qt}, \ldots, x_d^{qt})$ for all $q \gg 1$, and so $cr^q \in (x_1^{qt}, \ldots, x_d^{qt}) :_R (x_1 \cdots x_d)^{qt-q} = (x_1^q, \ldots, x_d^q)$ for all $q \gg 1$, since we may compute the colon ideal formally. This shows that $r \in I^*$, and so we have that $z \in (x_1^t, \ldots, x_d^t)R + (x_1 \cdots x_d)^{t-1}I^*$. But $uI^* \subseteq I$, and so $uz \in (x_1^t, \ldots, x_d^t)R + (x_1 \cdots x_d)^{t-1}I = (x_1^t, \ldots, x_d^t)R$, a contradiction. $\square$