In all of these problems assume that all given rings are Noetherian of characteristic p, unless otherwise specified. R always denotes a given ring, and I an ideal of R. As usual, $q = p^e$ with $e \in \mathbb{N}$.

1. Let S be an R-algebra with spanning set M and let T be an S-algebra with $N \subseteq T$. Suppose that $MN = \{\mu\nu : \mu \in M,\, \nu \in N\}$ is a free basis for T over R. Show that M is a free basis for S over R and that N is a free basis for T over S. (This was used tacitly in one of the proofs in class.)

2. Let $x_1, \ldots, x_n \in R$ and suppose that x_n is a nonzerodivisor on the R-module M. Prove that $H_j(x_1, \ldots, x_n;\, M) \cong H_j(x_1, \ldots, x_{n-1};\, M/x_nM)$ for all j.

3. Let $x_1, \ldots, x_n$ be a system of parameters for an excellent reduced equidimensional local ring R such that every x_i is a test element. Let $I = (x_1, \ldots, x_n)R$ and $J = I^*$. Prove that $J^{[p]} \subseteq I^{[p]} + (x_1 \cdots x_n)^{p-1}J$. Thus, by a class result, $J \subseteq IS$ for some module-finite extension S of R.

4. Let J be an ideal of a Noetherian domain R. Prove that if $J^{[q]} \subseteq I^{[q]} + (I^{[q]} : I)J$ then $J \subseteq I^*$. (In his thesis, Ian Aberbach proved the very much harder result that this condition is sufficient to force $J \subseteq IS$ for some module-finite extension S of R.)

5. Let R be an excellent bi-equidimensional semilocal ring which is unmixed, i.e., such that there are no embedded primes. Let R have dimension d. Let $x_1, \ldots, x_k$ be part of a system of parameters in the Jacobson radical of R, and let $I = (x_1, \ldots, x_k)R$. Then R/I^{unmx} has the same properties that R does, except that the Krull dimension is $d - k$.