1. Show that if $A \to S \to T$ are rings with T faithfully flat over S and M is any R-module then $\mathrm{Tor}_i^A(M, T) \cong \mathrm{Tor}_i^A(M, S) \otimes T$. Conclude that the map $\mathrm{Tor}_i^A(M, S) \to \mathrm{Tor}_i^A(M, T)$ is injective.

2. Without consulting references,[1] prove that if $R \to S$ is a faithfully flat map with R Noetherian then a subset of $\mathrm{Spec}\, R$ is closed iff its inverse image in $\mathrm{Spec}\, S$ is closed.

[Here is a sketch involving several preliminary problems:

A subset of a topological space is called *locally closed* if it is the intersection of a closed set and an open set, *constructible* if it is a finite union of locally closed sets and *pro-constructible* if it is an intersection of constructible sets. A topological space is called *Noetherian* if the closed subsets satisfy DCC: $\mathrm{Spec}\, R$ for R Noetherian is obviously Noetherian.

(a) Let X be a Noetherian topological space. Then $Z \subseteq X$ is constructible iff $(*)$ for every non-empty irreducible closed set $X_0 \subseteq X$, either $Z \cap X_0$ is not dense in X_0 or $Z \cap X_0$ contains a non-empty open subset of X_0.

(b) If $f : Y \to X$ is the map of Specs induced by $R \to S$ with R Noetherian, them $f(Y)$ is pro-constructible in X (constructible if S is finitely generated over R: do this case first).

(c) Let $X = \mathrm{Spec}\, R$, where R is Noetherian. Then Z is closed in X iff it is pro-constructible and contains the closure of each of its points.

(d) Let $R \to S$ be such that R is Noetherian, and let $f : Y \to X$ be the induced map $\mathrm{Spec}\, S \to \mathrm{Spec}\, R$. Suppose that f is onto and that *going down* holds, i.e., if $Q \in \mathrm{Spec}\, S$, $P = f(Q)$, and $P_0 \in \mathrm{Spec}\, R$ is such that $P_0 \subseteq P$, then there is a $Q_0 \in \mathrm{Spec}\, S$ with $Q_0 \subseteq Q$ and $f(Q_0) = P_0$. Then $Z \in X$ is closed iff $f^{-1}(Z) \subseteq Y$ is closed. Show that these conditions hold when R is Noetherian and S is faithfully flat over R.]

3. Let $R \to S$ be any faithfully flat map of Noetherian rings. Suppose that the fibers are Cohen-Macaulay (respectively, Gorenstein). Show that if the non-Cohen-Macaulay (respectively, non-Gorenstein locus) in S is closed, then so is the non-Cohen-Macaulay (respectively, non-Gorenstein) locus in R. Show also that if I defines the non-Cohen-Macaulay locus in R, then IS defines the non-Cohen-Macaulay locus in S.

4. Let R be a domain, $W \subseteq R$ a multiplicative system and N a submodule of a free R-module G. Prove that $(W^{-1}R)^+ \cong W^{-1}(R^+)$ and that $W^{-1}(NR^+ \cap G) \cong W^{-1}N\big(W^{-1}(R^+)\big) \cap W^{-1}G$ (i.e., plus closure of submodules commutes with localization). (NR^+ is the span of N in $R^+ \otimes_R G \supseteq G$.)

5. Let K be a field of char. $p \neq 3$, 0, and let $R = K[x, y, z] = K[X, Y, Z]/(X^3+Y^3+Z^3)$. Construct a specific[2] module-finite extension S of R such that $z^2 \in (x, y)S$.

[1] [H. Matsumura, *Commutative Algebra*, W. A. Benjamin, Inc., New York, 1970], pp. 38–48, or [EGA].

[2] If $p \equiv 2 \bmod 3$, one can use a purely inseparable extension, but not if $p \equiv 1 \bmod 3$.