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TIGHT CLOSURE THEORY AND CHARACTERISTIC P METHODS

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0. Introduction

Unless otherwise specified, the rings that we consider here will be Noetherian rings R containing a field. Frequently, we restrict, for simplicity, to the case of domains finitely generated over a field K. The theory of tight closure exists in much greater generality, and we refer the reader to [Ab], [Bren1–2], [Bru], [Ha], [HaW1–2], [HaWY1–2], [HH1–14], [Ho1–3], [Hu1–2], [LySm], [Si1–9], [Sm1–5], and [Vr1–4] for the development of the larger theory as well as discussion of related topics such as the existence of big Cohen-Macaulay algebras.

One way of thinking about many closure operations is to view them as arising from necessary conditions for an element to be in an ideal. If the condition fails, the element is not in the ideal. If the condition is not both necessary and sufficient, then when it holds, the element might be in the ideal, but it may only be in some larger ideal, which we think of as a kind of closure.

Tight closure in positive characteristic can be thought of as arising from such a necessary but not sufficient condition for ideal membership. One of the reasons that it is so useful for proving theorems is that in some rings, the condition is both necessary and sufficient. In particular, that is true in regular rings. In consequence, many theorems can be proved about regular rings that are rather surprising. They have the following nature: one can see that in a regular ring a certain element is "almost" in an ideal. Tight closure permits one to show that the element actually is in the ideal. This technique works like magic on several major results that seemed very difficult before tight closure came along.

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One has to go to some considerable trouble to get a similar theory working in rings that contain the rationals, but this has been done, and the theory works extremely well for "nice" Noetherian rings like the ones that come up in algebraic and analytic geometry.

It is still a mystery how to construct a similar theory for rings that do not contain a field. This is not a matter of thinking about anything pathological. Many conjectures could be resolved if one had a good theory for domains finitely generated as algebras over the integers.

Before proceeding to talk about tight closure, we give some examples of necessary and/or sufficient conditions for membership in an ideal. The necessary conditions lead to a kind of closure.

(1) A necessary condition for $r \in R$ to be in the ideal I is that the image of r be in IK for every homomorphism of R to a field K. This is not sufficient: the elements that satisfy the condition are precisely the elements with a power in I, the *radical* of I.

(2) A necessary condition for $r \in R$ to be in the ideal I is that the image of r be in IV for every homomorphism of R to a valuation ring V. This is not sufficient: the elements that satisfy the condition are precisely the elements in $\overline{I}$, the *integral closure* of I. If R is Noetherian, one gets the same integral closure if one only considers Noetherian discrete valuation rings V. There are many alternative definitions of integral closure.

(3) If R has positive prime characteristic p let S_e denote R viewed as an R algebra via the eth iteration F^e of the Frobenius endomorphism F (thus, $S_e = R$, but the structural homomorphism $R \to S_e = R$ sends r to r^{p^e}). A necessary condition that $r \in I$ is that for some integer e, $r^{p^e} \in IS_e$. Note that when S_e is identified with R, IS_e becomes the ideal generated by all elements i^{p^e} for $i \in I$. This ideal is denoted $I^{[p^e]}$. This condition is not sufficient for membership in I. The corresponding closure operation is the Frobenius closure I^F of I: it consists of all elements $r \in R$ such that $r^{p^e} \in I^{[p^e]}$ for some nonnegative integer e. (Note: once this holds for once choice of e, it holds for all larger choices.) E.g., in $K[x, y, z] = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$, if K has characteristic 2 (quite explicitly, if $K = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$) then with $I = (x, y)$, we have that $z^2 \in I^F - I$. In fact, $(z^2)^2 \in I^{[2]} = (x^2, y^2)$ here, since $z^4 = z^3 z = -(x^3 + y^3)z \in (x^2, y^2)$.

Finally, here is a test for ideal membership that is sufficient but *not* necessary. It was used in the first proof of the Briançon-Skoda theorem, and we want to mention it, although easier proofs by analytic methods are available now.

(4) Skoda's analytic criterion. Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and ϕ a plurisubharmonic function[1] on Ω. Let f and $g_1, \dots, g_k$ be holomorphic functions on Ω. Let $\gamma = (|g_1|^2 + \cdots + |g_k|^2)^{1/2}$. Let X be the set of common zeros of the g_j. Let $d = \max\{n, k-1\}$. Let λ denote Lebesgue measure on $\mathbb{C}^n$. Skoda's criterion asserts that *if one has either that for some real $\alpha > 1$,*

$$\int_{\Omega - X} \frac{|f|^2}{\gamma^{2\alpha d+2}} e^{-\phi}\, d\lambda < +\infty,$$

or if one has that

$$\int_{\Omega - X} \frac{|f|^2}{\gamma^{2d}} \bigl(1 + \Delta \log(\gamma)\bigr) e^{-\phi}\, d\lambda < +\infty,$$

then there exist $h_1, \dots, h_k$ holomorphic on Ω such that $f = \sum_{j=1}^k h_j g_j$. Hilbert's Nullstellensatz states that if f vanishes at the common zeros of the g_j then $f \in \operatorname{Rad} I$ where $I = (g_1, \dots, g_k)$. The finiteness of any of the integrals above conveys the stronger information that, in some sense, f is "small" whenever all the g_j are "small" (or the integrand will be too "large" for the integral to converge), and we get the stronger conclusion that $g \in I$.

1. Reasons for thinking about tight closure

We give here five results valid in any characteristic (i.e., over any field) that can be proved using tight closure theory. The tight closure proofs are remarkably simple, at least in the main cases. The terminology used in the following closely related theorems is discussed briefly after their statements.

Theorem 1.1 (Hochster-Roberts). *Let S be a regular ring that is an algebra over the field K, and let G be a linearly reductive algebraic group over K acting on S. Then the ring of invariants $R = S^G$ is a Cohen-Macaulay ring.*

Theorem 1.1$^{\mathrm{O}}$ (Hochster-Huneke). *If R is a direct summand (as an R-module) of a regular ring S containing a field, then R is Cohen-Macaulay.*

Theorem (1.1$^{\mathrm{O}}$) implies Theorem (1.1). Both apply to many examples from classical invariant theory. Recall that an algebraic group (i.e., a Zariski closed subgroup of $GL(n,K)$) is called *linearly reductive* if every representation is completely reducible. In characteristic 0, these are the same as the reductive groups and include finite groups, products of $GL(1,K)$ (algebraic tori), and semi-simple groups. Over $\mathbb{C}$ such a group is

[1] We won't explain these terms from complex analysis here: the definitions are not so critical for us, because in the application to the Briançon-Skoda theorem, which we discuss later, we work in the ring of germs of holomorphic functions at, say, the origin in complex n-space, $\mathbb{C}^n$ ($\cong$ convergent power series $\mathbb{C}\{z_1, \dots, z_n\}$) — we can pass to a smaller, pseudoconvex neighborhood; likewise, ϕ becomes unimportant.

the complexification of compact real Lie group. A key point is that when a linearly reductive algebraic group acts on a K-algebra S, if S^G is the ring of invariants or fixed ring $\{s \in S : g(s) = s \text{ for all } g \in G\}$ there is a canonical retraction map map $S \to S^G$, called the *Reynolds operator*, that is S^G-linear. Thus, $R = S^G$ is a direct summand of S as an R-module.

In particular, if S is a polynomial ring over a field K and G is a linearly reductive linear algebraic group acting on S_1, the vector space of 1-forms of S, and, hence, all of S (the action should be an appropriate one, i.e., determined by a K-morphism of G into the automorphisms of the vector space S_1), then the fixed ring S^G is a Cohen-Macaulay ring R. What is a Cohen-Macaulay ring? The issue is local: for a local ring the condition means that some (equivalently, every) system of parameters is a regular sequence. In the graded case the Cohen-Macaulay condition has the following pleasant interpretation: when R is represented as a finitely generated module over a graded polynomial subring A , R is free over A. This is a very restrictive and useful condition on R, especially in higher dimension. The Cohen-Macaulay condition is very important in intersection theory. Notice that since moduli spaces are frequently constructed as quotients of smooth varieties by actions of reductive groups, Theorem 1.1 implies the Cohen-Macaulay property for many moduli spaces.

Theorem 1.1 was first proved by a complicated reduction to characteristic $p > 0$ [HR]. Boutot [Bou] gave a shorter proof for affine algebras in characteristic 0 using resolution of singularities and the Grauert-Riemenschneider vanishing theorem. The tight closure proof of Theorem 1.1$^{\mathrm{O}}$ is the simplest in many ways.

Theorem 1.2 (Briançon-Skoda theorem). *Let R be a regular ring and I an ideal of R generated by n elements. Then $\overline{I^n} \subseteq I$.*

We gave one characterization of what $u \in \overline{J}$ means earlier. It turns out to be equivalent to require that there be an equation

$$u^h + j_1 u^{h-1} + \cdots + j_h = 0$$

such that every $j_t \in J^t$, $1 \le t \le h$. We shall give a third characterization later.

Theorem 1.2 was first proved by analytic techniques: cf. (4) of the Introduction. See [BrS] and [Sk]: in the latter paper the analytic criteria needed were proved. The first algebraic proofs were given in [LT] (for a very important special case) and [LS].

Theorem 1.3 (Ein-Lazarsfeld-Smith comparison theorem). *Let P be a prime ideal of codimension h in a regular ring. Then $P^{(hn)} \subseteq P^n$ for every integer n.*

This was most unexpected. The original proof, valid in characteristic 0, ultimately depends on resolution of singularities and deep vanishing theorems. Cf. [ELS]. The tight closure proof [HH14] permits one to extend the results to characteristic p as well as recovering the characteristic 0 result.

Theorem 1.4 (Hochster-Huneke). *Let R be a reduced equidimensional finitely generated K-algebra, where K is algebraically closed. Let $f_1, \dots, f_h$ be elements of R that generate an ideal I of codimension (also called height) h mod every minimal prime of R. Let J be the Jacobian ideal of R over K. Then J annihilates the Koszul cohomology $H^i(f_1, \dots, f_h; R)$ for all $i < h$, and hence the local cohomology $H^i_I(R)$ for $i < h$.*

This result is a consequence of phantom homology theory, test element theory for tight closure, and the Lipman-Sathaye Jacobian theorem [LS], all of which we will describe eventually. If

$$R \cong K[x_1, \dots, x_n]/(f_1, \dots, f_m)$$

has codimension r in $\mathbb{A}^n_K$, then J is the ideal of R generated by the images of the size r minors of the Jacobian matrix $(\partial f_j/\partial x_j)$, and defines the non-smooth (over K) locus in Spec R. The ideal $J \subseteq R$ turns out to be independent of which presentation of R one chooses.

Here is a more geometrically flavored corollary.

Corollary 1.5. *Let R be a finitely generated graded domain of dimension $n + 1$ over an algebraically closed field K, so that $X = Proj(R)$ is a projective variety of dimension n over an algebraically closed field K. Let g denote a homogeneous element of the Jacobian ideal $J \subseteq R$ of degree d (so that g gives a global section of $\mathcal{O}_X(e)$. Then for $1 \le j \le n - 1$, the map $H^j(X, \mathcal{O}_X(t)) \to H^j(X, \mathcal{O}_X(t + e))$ induced by multiplication by g is 0.*

The reason that this is a Corollary of Theorem 1.4 is that for $j \ge 1$, if we let $M = \bigoplus_{t\in\mathbb{Z}} H^j(X, \mathcal{O}_X(t))$, then M is isomorphic (as an R-module) with $H^{j+1}_m(R)$ which may be viewed as an R-module. We may replace m by the ideal generated by a homogeneous system of parameters, since the two have the same radical. Then Theorem 4 implies that the Jacobian ideal of R kills M for $1 \le j \le n - 1$.

2. The definition of tight closure in positive characteristic p

One of our guidelines towards a heuristic feeling for when an element u of a Noetherian ring R should be viewed as "almost" in an ideal $I \subseteq R$ will be this: if R has a module-finite extension S such that $u \in IS$ then u is "almost" in I.

Notice that if R is a normal domain (i.e., integrally closed in its field of fractions) containing the rational numbers and S is a module-finite extension, then $IS \cap R = I$, so that for normal rings containing $\mathbb{Q}$ we are not allowing any new elements into the ideal. One can see this as follows. By first killing a minimal prime ideal of S disjoint from $R - \{0\}$ we may assume that S is a domain. Let $L \to L'$ be the corresponding finite algebraic extension of fraction fields, and suppose it has degree d. Let $\mathrm{tr}_{L'/L}$ denote field trace. Then $\dfrac{1}{d}\mathrm{tr}_{L'/L} : S \to R$ gives an R-linear retraction when R is normal. This implies that $IS \cap R = I$ for every ideal I of R. (We only need the invertibility of the single integer d in R for this argument.)

The situation for normal domains of positive characteristic is very different, where it is an open question whether the elements that are "almost" in an ideal in this sense may coincide with the tight closure in good cases. Our definition of tight closure may seem unrelated to the notion above at first, but there is a close connection.

For simplicity we start with the case of ideals in Noetherian domains of characteristic $p > 0$. Recall that in characteristic p the Frobenius endomorphism $F = F_R$ on R maps r to r^p, and is a ring endomorphism. When R is reduced, we denote by R^{1/p^e} the ring obtained by adjoining p^eth roots for all elements of R: it is isomorphic to R, using the eth iterate of its Frobenius endomorphism with the image restricted to R. Recall that in a ring of positive characteristic p, when $q = p^e$, we denote by $I^{[q]}$ the ideal of R generated by all qth powers of elements of I. It is easy to see that this ideal is generated by qth powers of generators of I. Notice that it is *much* smaller, typically, than the ordinary power I^q. I^q is generated by *all monomials* of degree q in the generators of I, not just qth powers of generators.

Definition 2.1. Let R be a Noetherian domain of characteristic $p > 0$ let I be an ideal of R and let u be an ideal of R. We say that $u \in R$ is in the *tight closure* I^* of I in R if there exists an element $c \in R - \{0\}$ such that for all sufficiently large $q = p^e$, we have that $cu^q \in I^{[q]}$.

It is equivalent in the definition above to say "for all q" instead of "for all sufficiently large q". We want to discuss why this condition should be thought of as placing u "almost" in I in some sense. Let $I = (f_1, \ldots, f_h)R$ Note that for every large $q = p^e$ one has

$$cu^q = r_{1q}f_1^q + \cdots r_{hq}f_h^q$$

and if we take qth roots we have

$$c^{1/q}u = r_{1q}^{1/q}f_1 + \cdots r_{hq}^{1/q}f_h,$$

an equation that holds in the ring $S_q = R[c^{[1/q]}, r_{iq}^{1/q} : 1 \le i \le h]$. S is a module-finite extension of R. But this is not quite saying that u is in IS: rather, it says that $c^{1/q}u$ is in IS. But for very large q, for heuristic purposes, one may think of $c^{1/q}$ as being close to 1: after all, the exponent is approaching 0. Thus, u is multiplied into IS_q in a sequence of module-finite extensions by elements that are getting closer and closer to being a unit, in a vague heuristic sense. This may provide some motivation for the idea that elements that are in the tight closure of an ideal are "almost" in the ideal.

It is ironic that tight closure is an extremely useful technique for proving theorems about regular rings, because it turns out that in regular rings the tight closure of any ideal I is simply I itself. In some sense, the reason that tight closure is so useful in regular rings is that it gives a criterion for being in an ideal that, on the face of it, is considerably weaker than being in the ideal. We shall return to this point later.

We may extend the definition to Noetherian rings R of positive prime characteristic p that are not necessarily integral domains in one of two equivalent ways:

(1) Define u to be in I^* if the image of u in R/P is in the tight closure of $I(R/P)$ in R/P for every minimal prime P of R.

(2) Define u to be in I^* if there is an element $c \in R$ and not in any minimal prime of R such that $cu^q \in I \diamond [q]$ for all $q = p^e \gg 0$.

3. Basic properties of tight closure and the Briançon-Skoda theorem

The following facts about tight closure in a Noetherian ring R of positive prime characteristic p are reasonably easy to verify from the definition.

(a) For any ideal I of R, $(I^*)^* = I^*$.

(b) For any ideals $I \subseteq J$ of R, $I^* \subseteq J^*$. We shall soon need the following characterization of integral closure of ideals in Noetherian domains.

Fact 3.1. *Let R be a Noetherian domain and let J be an ideal. Then $u \in R$ is in $\overline{J}$ if and only if for some $c \in R - \{0\}$ and every integer positive integer n, $cu^n \in I^n$. It suffices if $cu^n \in I^n$ for infinitely many values of n.*

Comparing this with the definition of tight closure and using the fact that $I^{[q]} \subseteq I^q$ for all $q = p^e$, we immediately get

(c) For any ideal I of R, $I^* \subseteq \overline{I}$. In particular, I^* is contained in the radical of I.

Less obvious is the following theorem that we will prove later.

Theorem 3.1. *If R is regular, every ideal of R is tightly closed.*

Assuming this fact for a moment, we can prove the following result:

Theorem 3.2 (tight closure form of the Briançon-Skoda theorem in characteristic p). *Let $I = (f_1, \dots, f_n)R$ be an ideal of a regular ring R of characteristic $p > 0$. Then $\overline{I^n} \subseteq I$. When R is not necessarily regular, it is still true that $\overline{I^n} \subseteq I^*$.*

Proof. Assuming Theorem 3.1 for the moment, we need only check the final assertion. It suffices to work modulo each minimal prime of R in turn, so we may assume that R is a domain. Then $u \in \overline{I^n}$ implies that for some nonzero c, $cu^m \in (I^n)^m$ for all m. Restricting $m = q = p^e$ we find that $cu^m \in I^{nq} \subseteq I^{[q]}$ for all q, since a monomial in n elements of degree nq must have a factor in which one of the elements is raised to the q th power. $\square$

Why is every ideal in a regular ring tightly closed? We first need the following:

Fact 3.3. *If R is regular of positive characteristic p then the Frobenius endomorphism is flat.*

This can be shown as follows: the issue is local on R. In the local case it suffices to prove it for the completion $\widehat{R}$ because $R \to \widehat{R}$ is faithfully flat. We have therefore reduced to considering the case $R \cong K[[x_1, \ldots, x_n]]$. The Frobenius map is then isomorphic with the ring inclusion $K^p[[x_1^p, \ldots, x_n^p]] \subseteq K[[x_1, \ldots, x_n]]$. Letting $K^p = k$, we may factor this map as $k[[x_1^p, \ldots, x_n^p]] \subseteq k[[x_1, \ldots, x_n]] \subseteq K[[x_1, \ldots, x_n]]$. The first extension is free on the monomials $x_1^{h_1} \cdots x_n^{h_n}$ with $0 \leq h_i < p$ for all i. The flatness of the second map (for any field inclusion $k \subseteq K$) may be seen as follows: since K is flat (in fact, free) over k, $K[x_1, \ldots, x_n]$ is flat over $k[x_1, \ldots, x_n]$. This is preserved when we localize at the maximal ideal generated by the x's in the larger ring and its contraction (also generated by the x's) to the smaller ring. Finally, it is further preserved when we complete both local rings. $\square$

Recall that for an ideal I of R and element $u \in R$, $I : u = \{r \in R : ur \in I\}$. This may thought of as the annihilator in R of the image of u in R/I.

Fact 3.4. *If $f : R \to S$ is flat, $I \subseteq R$ and $u \in R$, then $IS : f(u) = (I : u)S$.*

To see why, note the exact sequence $(I : u)/I \to R/I \to R/I$ where the map is multiplication by u. Applying $S \otimes_R$ preserves exactness, from which the stated result follows.

Corollary 3.5. *If R is regular of positive characteristic, I is any ideal, and $u \in R$, then $I^{[q]} : u^q = (I : u)^{[q]}$ for all $q = p^e$.*

The point is that since $F : R \to R$ is flat, so is its eth iterate F^e. If S denotes R viewed as an R-algebra via F^e then $IS = I^{[p^e]}$ when we "remember" that S is R. With this observation, Corollary 3.5 follows from Fact 3.4.

Proof of Theorem 3.1. We can reduce to the case where R is a domain. If $c \neq 0$ and $cu^q \in I^{[q]}$ for all $q = p^e$, then $c \in \bigcap_q I^{[q]} : u^q = \bigcap_q (I : u)^{[q]} \subseteq \bigcap_q (I : u)^q$. Since the intersection is not 0, we must have that $I : u = R$, i.e., that $u \in R$. $\square$

4. Direct summands of regular rings are Cohen-Macaulay

Elements $x_1, \ldots, x_n$ in a ring R are called a *regular sequence* on an R-module M if $(x_1, \ldots, x_n)M \neq M$ and x_{i+1} is not a zerodivisor on $M/(x_1, \ldots, x_i)M$, $0 \leq i < n$. A sequence of indeterminates in a polynomial or formal power series ring R, with $M = R$ (or a nonzero free R-module) is an example. We shall make use of the following fact:

Fact 4.1. *Let A be a polynomial ring over a field K, say $A = K[x_1, \ldots, x_d]$ or let A be a regular local ring in which $x_1, \ldots, x_d$ is a minimal set of generators of the maximal ideal.*

Then a finitely generated nonzero A-module M (assumed graded in the first case) is A-free if and only if $x_1, \dots, x_d$ is a regular sequence on M. Thus, a module-finite extension ring R (graded if A is a polynomial ring) of A is Cohen-Macaulay if and only if $x_1, \dots, x_d$ is a regular sequence on R.

The following two lemmas make the connection between tight closure and the Cohen-Macaulay property.

Proposition 4.2. *Let S be a module-finite domain extension of the domain R (torsion-free is sufficient) and let $x_1, \dots, x_d$ be a regular sequence in R. Suppose $0 \le k \le d$ and let $I = (x_1, \dots, x_k)R$. Then $IS :_S x_{k+1} \subseteq (IS)^*$ in S.*

Thus, if every ideal of S is tightly closed, and $x_1, \dots, x_d$ is a regular sequence in R, it is a regular sequence in S.

Proof. Because S is a torsion-free R-module there is an an element c of $R - \{0\}$ that multiplies S into an R-free submodule $G \cong R^h$ of S. (This is really all we need about S.) Suppose that $ux_{k+1} \in IS$. Raise both sides to the $q = p^e$ power to get $u^q x_{k+1}^q \in I^{[q]}S$. Multiply by c to get $(cu^q)x_{k+1}^q \in I^{[q]}G$. Because the x_j form a regular sequence on G, so do their q th powers, and we find that $cu^q \in I^{[q]}S = (IS)^{[q]}$. Since this holds for all $q = p^e$, we are done. $\square$

Proposition 4.3. *Let R be a domain module-finite over a regular local ring A or $\mathbb{N}$-graded and module-finite over a polynomial ring A. Suppose that R is a direct summand of a regular ring S as an R-module. Then R is Cohen-Macaulay (i.e., A-free).*

Proof. Let $x_1, \dots, x_d$ be as in Fact 4.1. The result comes down to the assertion that $x_1, \dots, x_d$ is a regular sequence on R. By Proposition 4.2, it suffices to show that every ideal of R is tightly closed. But if J is an ideal of R and $u \in R$ is in J^*, then it is clear that $u \in (JS)^* = JS$, since S is regular, and so $u \in JS \cap R = J$, because R is a direct summand of S. $\square$

Pushing this idea a bit further, one gets a full proof of Theorem 1.1. We need to extend the notion of tight closure to equal characteristic 0, however. This is tackled in §6.

5. The Ein-Lazarsfeld-Smith comparison theorem

We give here the characteristic p proof of (1.3), and we shall even allow radical ideals, with h taken to be the largest height of any minimal prime. For a prime ideal P, $P^{(N)}$, the N th symbolic power, is the contraction of $P^N R_P$ to R. When I is a radical ideal with minimal primes $P_1, \dots, P_k$ and $W = R- = \bigcup_j P_j$, we may define $P^{(N)}$ either as $\bigcap_j P_j^{(N)}$ or as the contraction of $I^N(W^{-1}R)$ to R.

Suppose that $I \neq (0)$ is radical ideal. If $u \in I^{(hn)}$, then for every $q = p^e$ we can write $q = an + r$ where $a \ge 0$ and $0 \le r \le n-1$ are integers. Then $u^a \in I^{(han)}$ and

$I^{hn}u^a \subseteq I^{hr}u^a \subseteq I^{(han+hr)} = I^{(hq)}$. We now come to a key point: we can show that $(*)$ $I^{(hq)} \subseteq I^{[q]}$. To see this, note that because the Frobenius endomorphism is flat for regular rings, $I^{[q]}$ has no associated primes other than the minimal primes of I, and it suffices to check $(*)$ after localizing at each minimal prime P of I. But after localization, I has at most h generators, and so each monomial of degree hq in these generators is a multiple of the q th power of at least one of the generators. This completes the proof of $(*)$. Taking n th powers gives that $I^{hn^2}u^{an} \subseteq (I^{[q]})^n = (I^n)^{[q]}$, and since $q \geq an$, we have that $I^{hn^2}u^q \subseteq (I^n)^{[q]}$ for fixed h and n and all q. Let d be any nonzero element of I^{hn^2}. The condition that $du^q \in (I^n)^{[q]}$ for all q says precisely that u is in the tight closure of I^n in R. But in a regular ring, every ideal is tightly closed, and so $u \in I^n$, as required. $\square$

6. Extending the theory to affine algebras in characteristic 0

In this section we discuss briefly how to extend the results of tight closure to finitely generated algebras over a field K of characteristic zero. There is a good theory with essentially the same properties as in positive characteristic.

Suppose that we have a finitely generated K-algebra R. We may think of R as having the form $K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ for finitely many polynomials f_j. An ideal $I \subseteq R$ can be given by specifying finitely many polynomials $g_j \in T = K[x_1, \ldots, x_n]$ that generate it, and an element u of R can be specified by giving a polynomial h that maps to. We can then choose a finitely generated $\mathbb{Z}$-subalgebra B of K that contains all of the coefficients of the f_j, the g_j and of h. We can form a ring $R_B = B[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ and we can consider the ideal I_B of the R_B generated by the images of the g_j in R_B. It turns out that after localizing B at one nonzero element we can make other pleasant assumptions: that $I_B \subseteq R_B \subseteq R$, that R_B and R_B/I_B are B-free (the lemma of generic freeness), and that tensoring with K over B converts $I_B \subseteq R_B$ to $I \subseteq R$. Moreover, h has an image in $R_B \subseteq R$ that we may identify with u.

We then define u to be in the tight closure of I in R provided that for all maximal Q in a dense open subset of the maximal spectrum of B, with $\kappa = B/Q$, the image of u in $R_\kappa = \kappa \otimes_B R_B$ is in the characteristic p tight closure of $I_\kappa = IR_\kappa$ — this makes sense because B/Q will be a finite field.

This definition turns out to be independent of the choices of B R_B, I_B, etc.

Here is one very simple example. Let $R = K[x,y,z]/(x^3+y^3+z^3)$ where K is any field of characteristic 0, e.g., the complex numbers, let $I = (x,y)$ and u be the image of z^2. In this case we may take $B = \mathbb{Z}$, $R_{\mathbb{Z}} = \mathbb{Z}[x,y,z]/(x^3+y^3+z^3)$ and $I_Z = (x,y)R_{\mathbb{Z}}$. Then z^2 is in the characteristic 0 tight closure of $(x,y)R$ because for every prime integer $p \neq 3$ (these correspond to the maximal ideals of $\mathbb{Z}$, the image of z^2 is in the characteristic

p tight closure of $(x, y)(\mathbb{Z}/p\mathbb{Z})[x, yz]/(x^3 + y^3 + z^3)$. Take $c = x$, for example. One can check that $c(z^2)^q \in (x^q, y^q)(R_/Z/pR_/Z)$ for all $q = p^e$. Write $2q = 3k + a$, $a \in \{1, 2\}$, and use that $xz^2q = \pm x(x^3 + y^3)kz^a$. Each term in $x(x^3 + y^3)^k$ has the form $x^{3i+1}y^{3j}$ where $3i + 3j = 3k \geq 2q - 2$. Since $(3i + 1) + 3j \geq 2q - 1$, at least one of the exponents is $\geq q$.

7. Test elements

In this section we again study the case of rings of characteristic $p > 0$. Let R be a Noetherian domain. We shall say that an element $c \in R - \{0\}$ is a *test element* if for every ideal I of R, $cI^* \subseteq I$. An equivalent condition is that for every ideal I and element u of R, $u \in I^*$ if and only if $cu^q \in I^{[q]}$ for every $q = p^e \geq 1$. The reason that this holds is the easily verified fact that if $u \in I^*$, then $u^q \subseteq (I^{[q]})^*$ for all q. Thus, an element that is known to be a test element can be used in all tight closure tests. *A priori* the element used in tight closure tests for whether $u \in I^*$ in the definition of tight closure can vary with both I and u. The test elements together with 0 form an ideal called the *test ideal.*

Test elements are known to exist for domains finitely generated over a field. Any element $d \neq 0$ such that R_d is regular turns out to have a power that is a test element. We won't prove this here.

We will explain, however, why the Jacobian ideal of a domain finitely generated over an algebraically closed field is contained in the test ideal, which is one of the ingredients of Theorem 1.4.

Here is a useful result that leads to existence theorems for test elements.

Theorem 7.1. *Let R be a Noetherian domain module-finite over a regular domain A of characteristic $p > 0$, and suppose that the extension of fraction fields is separable. Then:*
(a) There are elements $d \in A - \{0\}$ such that $dR^{1/p} \subseteq R[A^{1/p}]$.
(b) For any d as in part (a), if $c = d^2$ then $(*)$ $cR^{1/q} \subseteq R[A^{1/q}$ for all q. Let $R_q = R[A^{1/q}]$.
(c) Any element $c \neq 0$ of R that satisfies the condition $(*)$ of part (b) is a test element for R.

Thus, R has test elements.

Proof. If we localize at all nonzero elements of A we are in the case where A is a field and R is a separable field extension. This is well-known and is left as an exercise for the reader. It follows that $R^{1/p}/R[A^{1/p}]$, which we may think of as a finitely generated $A^{1/p}$-module, is a torsion module. But then it is killed by an element of $A^{1/p} - \{0\}$ and, hence, by an element of $A - \{0\}$.

For part (b) note we note that since $dR^{1/p} \subseteq R[A^{1/p}]$, we have that $d^{1/q}R^{1/pq} \subseteq R^{1/q}[A^{1/pq}]$ for all $q = p^e$. Thus,

$$d^{1+1/p}R^{1/p^2} \subseteq d(d^{1/p}R^{1/p^2}) \subseteq d(R^{1/p}[A^{1/p^2}]) \subseteq R[A^{1/p}][A^{1/p^2}] = R[A^{1/p^2}].$$

Continuing in this way, one concludes easily by induction that

$$d^{1+1/p+\cdots+1/p^{e-1}}R^{1/p^e} \subseteq R[A^{1/p^e}].$$

Since $2 > 1 + 1/p + \cdots 1/p^{e-1}$ for all $p \geq 2$, we obtain the desired result.

Finally, suppose that c satisfies condition $(*)$. It suffices to show that for all I and $u \in I^*$, that $cu \in I$. But if $u \in I^*$ we can choose $a \in A - \{0\}$ (all nonzero elements of R have nonzero multiples in A) such that $au^q \in I^{[q]}$ for all $q = p^e$. Taking q th roots gives $a^{1/q}u \in IR^{1/q}$ for all q. Multiplying by c gives that $a^{1/q}cu \in IR[A^{1/q}] = IR_q$ for all q, and so $a^{1/q} \in IR_q :_{R_q} cu$ for all q. It is not hard to show that $R \otimes_A A^{1/q} \cong R[A^{1/q}]$ here. (The obvious map is onto, and since R is torsion-free over A and $A^{1/q}$ is A-flat, $R \otimes_A A^{1/q}$ is torsion-free over, so that we can check injectivity after localizing at $A - \{0\}$, and we thus reduce to the case where A is a field and R is a finite separable extension field, where the result is the well-known linear disjointness of separable and purely inseparable field extensions.) The flatness of Frobenius for A means precisely that $A^{1/q}$ is flat over A, so that R_q is flat over R; this is simply a base change. Thus, $IR_q : cu = (I :_R cu)R_q \subseteq (I :_R cu)R^{1/q}$. Hence, for all $q = p^e$, $a^{1/q} \in JR^{1/q}$, where $J = I :_R cu$. This shows that $a \in J^{[q]}$ for all q. Since $a \neq 0$, we must have that J is the unit ideal, i.e., that $cu \in I$.

The same argument works essentially without change when I is a submodule of a free module instead of an ideal. □

8. Test elements using the Lipman-Sathaye theorem

This section describes some material from Section (1.4) of [HH12].

For the moment, we do not make any assumption on the characteristic. Let $T \subseteq R$ be a module-finite extension, where T is a Noetherian domain, R is torsion-free as a T-module and the extension is generically smooth. Thus, if $\mathcal{K}$ is the fraction field of T and $\mathcal{L} = \mathcal{K} \otimes_T R$ is the total quotient ring of R then $\mathcal{K} \to \mathcal{L}$ is a finite product of separable field extensions of $\mathcal{K}$. The *Jacobian ideal* $\mathcal{J}(R/T)$ is defined as the 0 th Fitting ideal of the R-module of Kähler R-differentials $\Omega_{R/T}$, and may be calculated as follows: write $R \cong T[X_1, \ldots, X_n]/P$ and then $\mathcal{J}(R/T)$ is the ideal generated in R by the images of all the Jacobian determinants $\partial(g_1, \ldots, g_n)/\partial(X_1, \ldots, X_n)$ for n-tuples $g_1, \ldots, g_n$ of elements of P. Moreover, to generate $\mathcal{J}(R/T)$ it suffices to take all the n-tuples of g_i from a fixed set of generators of P.

Now suppose in addition that T is regular. Let R' be the integral closure of R in $\mathcal{L}$, which is well known to be module-finite over T (the usual way to argue is that any discriminant multiplies it into a finitely generated free T-module: cf. (2.4.5), part (g)). Let $J = \mathcal{J}(R/T)$ and $J' = \mathcal{J}(R'/T)$. The result of Lipman and Sathaye ([LS], Theorem 2, p. 200) may be stated as follows:

(8.1) Theorem (Lipman-Sathaye). *With notation as above (in particular, there is no assumption about the characteristic, and T is regular), suppose also that R is an integral domain. If $u \in \mathcal{L}$ is such that $uJ' \subseteq R'$ then $uJR' \subseteq R$. In particular, we may take $u = 1$, and so $JR' \subseteq R$.* $\square$

This property of "capturing the normalization" will enable us to produce test elements.

(8.2) Corollary (existence of test elements via the Lipman-Sathaye theorem). *If R is a domain module-finite over a regular domain A of characteristic p such that the extension of fraction fields is separable, then every element c of $J = \mathcal{J}(R/A)$ is such that $cR^{1/q} \subseteq A^{1/q}[R]$ for all q, and, in particular, $cR^\infty \subseteq A^\infty[R]$. Thus, if $c \in J \cap (R - \{0\})$, it is a test element.*

Proof. Since $A^{1/q}[R] \cong A^{1/q} \otimes_A R$, the image of c is in $\mathcal{J}(A^{1/q}[R]/A^{1/q})$, and so the Lipman-Sathaye theorem implies that c multiplies the normalization S of $A^{1/q}[R]$ into $A^{1/q}[R]$. Thus, it suffices to see that $R^{1/q}$ is contained in S. Since it is clearly integral over $A^{1/q}[R]$ (it is obviously integral over R), we need only see that the elements of $R^{1/q}$ are in the total quotient ring of $A^{1/q}[R]$, and for this purpose we may localize at $A^{\mathrm{o}} = A - \{0\}$. Thus, we may replace A by its fraction field and assume that A is a field, and then R is replaced by $(A^{\mathrm{o}})^{-1}R$, which is a separable field extensions. Thus, we come down to the fact that if $A \subseteq R$ is a finite separable field extension, then the injection $A^{1/q} \otimes_A R \to R^{1/q}$ (the map is an injection because separable and purely inseparable field extensions are linearly disjoint) is an isomorphism, which is immediate by a degree argument. $\square$

(8.3) Corollary (more test elements via the Lipman-Sathaye theorem). *Let K be a field of characteristic p and let R be a d-dimensional geometrically reduced (i.e., the ring stays reduced even when one tensors with an inseparable extension of K — this is automatic if K is perfect) domain over K that is finitely generated as a K-algebra. Let $R = K[x_1, \dots, x_n]/(g_1, \dots, g_r)$ be a presentation of R as a homomorphic image of a polynomial ring. Then the $(n-d) \times (n-d)$ minors of the Jacobian matrix $(\partial g_i/\partial x_j)$ are contained in the test ideal of R, and remain so after localization and completion. Thus, any element of the Jacobian ideal generated by all these minors that is in $R - \{0\}$ is a test element.*

Proof. We pass to $K(t) \otimes_K R$, if necessary, where $K(t)$ is a simple transcendental extension of K, to guarantee that the field is infinite. Our hypothesis remains the same, the Jacobian matrix does not change, and, since $K(t) \otimes_K R$ is faithfully flat over R, it suffices to consider the latter ring. Thus, we may assume without loss of generality that K is infinite. The calculation of the Jacobian ideal is independent of the choice of indeterminates. We are therefore free to make a linear change of coordinates, which corresponds to choosing an element of $G = GL(n, K) \subseteq K^{n^2}$ to act on the one-forms of $K[x_1, \dots, x_n]$. For a dense

Zariski open set U of $G \subseteq K^{n^2}$, if we make a change of coordinates corresponding to an element $\gamma \in U \subseteq G$ then, for every choice of d of the (new) indeterminates, if A denotes the K-subalgebra of R that these d new indeterminates generate, the two conditions listed below will hold:

(1) R will be module-finite over A (and the d chosen indeterminates will then, per force, be algebraically independent) and

(2) R will be generically smooth over A.

We may consider these two statements separately, for if each holds for a dense Zariski open subset of G we may intersect the two subsets. The first statement follows from the standard "linear change of variable" proofs of the Noether normalization theorem for affine K-algebras (these may be used whenever the ring contains an infinite field). For the second, we want each d element subset, say, after renumbering, $x_1, \ldots, x_d$, of the variables to be a separating transcendence basis for the fraction field L of R over K. (The fact that R is geometrically reduced over K implies that L is separably generated over K.) By, for example, either Theorem 5.10 (d) of [Ku] or Proposition 5.4 of [Swan] a necessary and sufficient condition for $x_1, \ldots, x_d$ to be a separating transcendence basis is that the differentials of these elements $dx_1, \ldots, dx_d$ in $\Omega_{L/K} \cong L^d$ be a basis for $\Omega_{L/K}$ as an L-vector space. Since the differentials of the original variables span $\Omega_{L/K}$ over L, it is clear that the set of elements of G for which all d element subsets of the new variables have differentials that span $\Omega_{L/K}$ contains a Zariski dense open set.

Now suppose that a suitable change of coordinates has been made, and, as above, let A be the ring generated over K by some set of d of the elements x_i. Then the $n-d$ size minors of $(\partial g_i/\partial x_j)$ involving the $n-d$ columns of $(\partial g_i/\partial x_j)$ that correspond to variables not chosen as generators of A precisely generate $\mathcal{J}(R/A)$. R is module-finite over A by the general position argument, and since it is equidimensional and reduced, it is likewise torsion-free over A, which is a regular domain. It is generically smooth likewise, because of the general position of the variables. The result is now immediate from (8.2): as we vary the set of d variables, every $n-d$ size minor occurs as a generator of some $\mathcal{J}(R/A)$ $\square$

9. Tight closure for submodules

In this section we make some brief remarks on how to extend the theory of tight closure to submodules of arbitary modules.

Let R be a Noetherian ring of positive prime characteristic p and let G be a free R-module with a specified free basis u_j, which we allow to be infinite. Then we may define an action of the Frobenius endomorphism F and its iterates on G very simply as follows: if $g = \sum_{i=1}^{t} r_i u_{j_i}$ (where the j_i are distinct) we let $F^e(g)$, which we also denote g^{p^e}, be

$\sum_{i=1}^{t} r_i^{p^e} u_{j_i}$. Thus, we are simply letting F act (as it does on the ring) on all the coefficients that occur in the representation of an element of G in terms of the free basis. If $N \subseteq G$ is a submodule, we let $N^{[p^e]}$ denote the submodule of G spanned by all the elements g^{p^e} for $g \in N$. We then define an element $x \in G$ to be in N^* if there exists $c \in R^{\text{o}}$ such that $cx^{p^e} \in N^{[p^e]}$ for all $e \gg 0$.

More generally, if M is any R-module, N is a submodule, and we want to determine whether $x \in M$ is in the tight closure N^* of N in M, we can proceed by mapping a free module G onto M, taking an element $g \in G$ that maps to x, letting H be the inverse image of N in G, and letting x be in N^*_M precisely when $g \in H^*_G$, where we are using subscripts to indicate the ambient module. This definition turns out to be independent of the choice of free module G mapping onto M, and of the choice of free basis for G.

I believe that there are many important questions about the behavior of tight closure for modules that are not finitely generated over the ring, especially for Artinian modules over local rings. See (3) of the next section.

However, for the rest of this section we restrict attention to the case of finitely generated modules. The theory of test elements for tight closure of ideals extends without change to the generality of modules.

In order to prove the result of Theorem 1.4 one may make use of a version of the phantom acyclicity theorem. We first recall the result of [BE] concerning when a finite free complex over a Noetherian ring R is acyclic. Suppose that the complex is

$$0 \to R^{b_n} \to \cdots \to R^{b_0} \to 0$$

and that r_i is the (determinantal) rank of the matrix α_i giving the map from $R^{b_i} \to R^{b_{i-1}}$, $0 \le i \le n+1$. The result of [BE] is that the complex is acyclic if and only if

(1) for $0 \le i \le n$, $b_i = r_{i+1} + r_i$

(2) for $1 \le i \le n$, the depth of the ideal J_i generated by the r_i size minors of α_i is at least i (this is automatic if the ideal generated by the minors is the unit ideal; by convention, the unit ideal has depth $+\infty$).

A complex $0 \to G_n \to \cdots \to G_0 \to 0$ is said to be *phantom acyclic* if for all $i \ge 1$, one has that the kernel Z_i of $G_i \to G_{i-1}$ is in the tight closure of the module of boundaries B_i (the image of G_{i+1} in G_i) in G_i. Note that this implies that Z_i/B_i is killed by the test ideal.

Now suppose that R is a reduced biequidimensional K-algebra, where K is a field of characteristic p. The phantom acyclicity criterion asserts (actually, in much greater generality: cf. [HH4, HH8] that a finite free complex as above is phantom acyclic provided that (1) holds and (2) holds under the weaker condition that every J_i have height at least i modulo every minimal prime. Thus, in a domain, this simply says that every J_i has height at least i: this replaces the subtle and difficult notion of "depth" by the much more tractable notion of "height" (or "codimension").

Theorem 1.4 is simply the result of applying the phantom acyclicity criterion to a Koszul complex. The conditions (1) and (2) are easy to verify. Therefore, the higher homology is killed by the test ideal, which contains the Jacobian ideal.

Further thoughts and questions

What we have said about tight closure so far is only the tip of an iceberg. Here are some major open questions.

1. Does tight closure commute with localization under mild assumptions on the ring? This is not known to be true even for finitely generated algebras over a field.

2. Under mild conditions, if a ring has the property that every ideal is tightly closed, does that continue to hold when one localizes? This is not known for finitely generated algebras over a field, nor for complete local rings. An affirmative answer to 1. would imply and affirmative answer to 2.

Rings such that every ideal is tightly closed are called *weakly F-regular*. The word "weakly" is omitted if this property also holds for all localizations of the ring. Weakly F-regular rings are Cohen-Macaulay and normal under very mild conditions — this holds even if one only assumes that ideals generated by parameters are tightly closed (this weaker property is called *F-rationality* and is closely related the notion of rational singularities; cf. [Hara], [Sm1,3]).

3. Let M be an Artinian module over, say, a complete reduced local ring with a perfect residue field. Let N be a submodule of M, Is it true that $u \in N^*_M$ if and only if there exists Q with $N \subseteq Q \subseteq M$ with Q/N of finite length such that $u \in N^*_Q$? This is true in a graded version and for isolated singularities [LySm1,2].

For any domain R, let R^+ denote the integral closure of R in an algebraic closure of its fraction field. This is unique up to non-unique isomorphism, and may be thought of as a "largest" domain extension of R that is integral over R.

4. For an excellent local domain R, is an element $r \in R$ in the tight closure of I if and only if it is in IS for some module-finite extension domain of R? It is equivalent to assert that for such a local domain R, $I^* = IR^+ \cap R$. This is known for ideals generated by part of a system of parameters: cf. [Sm1,2]. It is known that $IR^+ \cap R \subseteq I^*$.

It is known in characteristic p that for a complete local domain R, and element $u \in R$ is in I^* if and only if it is in $IB \cap R$ for some big Cohen-Macaulay algebra extension ring B of R: cf. §11 of [Ho1].

It is worth mentioning that there is an intimate connection between tight closure and the existence of big Cohen-Macaulay algebras B over local rings (R, m), i.e., algebras B

such that $mB \neq B$ and every system of parameters for R is a regular sequence on B. Tight closure ideas led to the proof in [HH7] that if R is an excellent local domain of characteristic p then R^+ is a big Cohen-Macaulay algebra. Moreover, for complete local rings R, it is known [Ho1], §11, that $u \in I^*$ if and only if R has a big Cohen-Macaulay algebra B such that $u \in IB$.

5. Is there an effective way to compute tight closures? The answer is not known even for ideals of cubical cones, i.e., of rings of the form $K[X, Y, Z]/(X^3+Y^3+Z^3)$ in positive characteristic different from 3. However, in cones over elliptic curves, tight closure agrees with plus closure for *homogeneous ideals*: cf. [Bren]. For ideals that are not homogeneous, the question raised in 4. is open even for such rings. When the characteristic of K is congruent to 2 mod 3, it is even possible that tight closure agrees with Frobenius closure in these rings. See [McD] and [Vr].

6. How can one extend tight closure to mixed characteristic? By far the most intriguing result along these lines is due to Ray Heitmann [Heit], who has proved that if (R, m) is a complete local domain of dimension 3 and mixed characteristic p, then every Koszul relation on parameters in R^+ is annihilated by multiplication by arbitrarily small positive rational powers of p (i.e., by $p^{1/N}$ for arbitrarily large integers n). This implies that regular local rings of dimension 3 are direct summands of their module-finite extension rings. Heitmann's result can be used to prove the existence of big Cohen-Macaulay algebras in dimension 3: cf. [Ho4].

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