HOMOLOGICAL INVARIANTS OF MODULES OVER CONTRACTING ENDOMORPHISMS

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Abstract. It is proved that when R is a local ring of positive characteristic, $\phi\colon R \to R$ is its Frobenius endomorphism, and some non-zero finite R-module has finite flat dimension or finite injective dimension for the R-module structure induced through ϕ, then R is regular. This broad generalization of Kunz's characterization of regularity in positive characteristic is deduced from a theorem concerning a local ring R with residue field of k of arbitrary characteristic: If ϕ is a contracting endomorphism of R, then the Betti numbers and the Bass numbers over ϕ of any non-zero finitely generated R-module grow at the same rate, on an exponential scale, as the Betti numbers of k over R.

1. Introduction

Given an endomorphism $\phi\colon R \to R$ of a commutative Noetherian local ring, each R-module M defines a module ${}^{\phi}M$: it has the same underlying additive group as M, and R acts on it by the rule $r \cdot m = \phi(r)m$. We study homological properties of ${}^{\phi}M$ when ϕ is *contracting*; this means that for each r in the maximal ideal $\mathfrak{m}$ of R the sequence $(\phi^i(r))_{i\geq 1}$ converges to 0 in the $\mathfrak{m}$-adic topology.

An R-module is said to be finite if it is finitely generated over R. We prove:

Theorem 1.1. *Let R be a local ring and $\phi\colon R \to R$ a contracting endomorphism.*

If there exist a finite non-zero R-module M and an integer $i \geq 1$, such that ${}^{\phi^i}M$ has finite flat dimension or finite injective dimension, then R is regular.

When the ring R has characteristic $p > 0$ and ϕ is the Frobenius map, $r \mapsto r^p$, the theorem implies that if ${}^{\phi}M$ is flat, then R is regular. We give a second, independent argument for this statement. Even when ${}^{\phi}M$ is free, it yields a substantial strengthening of the classical result of Kunz, [5, 2.1], which treats the case $M = R$. Comparison with other results is given in Remarks 5.2 and 5.4.

Other naturally occurring contracting endomorphisms are described in Section 5. Here we note that if there is a homomorphism of rings $R/\mathfrak{m} \to R$, which composed with the natural surjection $R \to R/\mathfrak{m}$ gives the identity of $R/\mathfrak{m}$, then the composition of these maps in reverse order is a contracting endomorphism of R. Extremal as it is, this example captures three motifs that run through the paper: Contracting endomorphisms exist in all characteristics; see Example 5.10. They exist only for

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equicharacteristic rings; see Remark 5.9. When seen through a contracting homomorphism, homological properties of finite R-modules mirror those of $k = R/\mathfrak{m}$.

The proof of Theorem 1.1 gives substance to the last point. It is obtained as a limit case of a result that establishes, in precise quantitative terms, that for *every* non-zero M over *any* R the (co)homology of ${}^{\phi}M$ behaves asymptotically as that of k. In particular, we characterize complete intersections in parallel with regular rings.

Indeed, let $\ell_R(-)$ denote length over R, and define the *curvature* of k by

$$\operatorname{curv}_R k = \limsup_n \sqrt[n]{\ell_R \operatorname{Tor}_n^R(k,k)}\,.$$

It measures, on an exponential scale, the asymptotic rate of growth of a minimal free resolution of k. All groups $\operatorname{Tor}_n^R(k, {}^{\phi}M)$ and $\operatorname{Ext}_R^n(k, {}^{\phi}M)$ have actions of R induced by the *original* action on the additive group shared by M and ${}^{\phi}M$. The resulting R-modules are annihilated by the ideal $\phi(\mathfrak{m})R$, and are finite when M is.

A version of our main theorem can now be stated as follows:

Theorem 1.2. *If $(R,\mathfrak{m})$ is a local ring, $\phi\colon R \to R$ a contracting endomorphism, and the ring $R/\phi(\mathfrak{m})R$ is artinian, then every finite non-zero R-module M satisfies*

$$\limsup_n \sqrt[n]{\ell_R \operatorname{Tor}_n^R(k, {}^{\phi}M)} = \operatorname{curv}_R k = \limsup_n \sqrt[n]{\ell_R \operatorname{Ext}_R^n(k, {}^{\phi}M)}\,.$$

The preceding results are corollaries of Theorem 5.1, where M is a complex with finite homology and the ring $R/\phi(R)$ is not assumed artinian. Absent the latter hypothesis, the numbers $\ell_R \operatorname{Tor}_n^R(k, {}^{\phi}M)$ and $\ell_R \operatorname{Ext}_R^n(k, {}^{\phi}M)$ need not be finite.

To deal with with this problem we replace lengths with Betti numbers and Bass numbers *over the map* ϕ. The definition of these numbers, given in Section 2, involves suitable Koszul complexes. This approach originates in [2], where it was developed for bounded complexes with finite homology. However, that context is too narrow to accommodate the proof of Theorem 5.1, even when M is an R-module. In Sections 3 and 4 we prove the relevant properties of homological invariants over ϕ, for complexes belonging to appropriate derived categories of R-modules.

Section 6 can be read independently of the preceding ones. Using tight closure methods, see [4], we give a different proof that a ring R of positive characteristic is regular if ${}^{\phi^i}M$ is flat for a finite module $M \neq 0$ and the Frobenius endomorphism ϕ.

2. Asymptotic invariants

Let R be commutative ring, $\mathsf{D}(R)$ the derived category of R-modules and Σ the translation functor; $\simeq$ flags isomorphisms in $\mathsf{D}(R)$. Complexes carry lower gradings:

$$M = \quad \cdots \longrightarrow M_{n+1} \xrightarrow{\partial_{n+1}^M} M_n \xrightarrow{\partial_n^M} M_{n-1} \longrightarrow \cdots$$

Let $\mathsf{D}_+^{\mathsf{f}}(R)$ (respectively, $\mathsf{D}_-^{\mathsf{f}}(R)$) denote the full subcategory of $\mathsf{D}(R)$ consisting of those complexes M for which the R-module $\operatorname{H}_n(M)$ is finite for each n, and is zero for $n \ll 0$ (respectively, $n \gg 0$). Set $\mathsf{D}_{\mathsf{b}}^{\mathsf{f}}(R) = \mathsf{D}_+^{\mathsf{f}}(R) \cap \mathsf{D}_-^{\mathsf{f}}(R)$. Modules are identified with complexes concentrated in degree 0, and the category of R-modules is identified with the full subcategory of $\mathsf{D}(R)$ with objects $\{M \in \mathsf{D}(R) \mid \operatorname{H}_n(M) = 0 \text{ for } n \neq 0\}$.

The derived functors of tensor products and of homomorphisms are denoted $- \otimes_R^{\mathsf{L}} -$ and $\mathsf{R}\mathrm{Hom}_R(-,-)$, respectively. For each integer n, we set

$$\operatorname{Tor}_n^R(-,-) = \operatorname{H}_n(- \otimes_R^{\mathsf{L}} -) \qquad \text{and} \qquad \operatorname{Ext}_R^n(-,-) = \operatorname{H}_{-n}(\mathsf{R}\mathrm{Hom}_R(-,-))$$

2.1. Let $\varphi\colon R \to S$ be a homomorphism of commutative Noetherian rings.

Complexes of S-modules are always viewed as complexes of R-modules by restricting scalars along φ. As explained in [2, 1.1], when M and N are complexes of S-modules the functors $-\otimes^{\mathsf{L}}_R M$ and $\mathsf{RHom}_R(-,N)$ induce functors

$$-\otimes^{\mathsf{L}}_R M\colon \mathsf{D}(R)\to\mathsf{D}(S) \quad\text{and}\quad \mathsf{RHom}_R(-,N)\colon \mathsf{D}(R)\to\mathsf{D}(S)\,,$$

When L is in $\mathsf{D}^{\mathsf{f}}_{+}(R)$ with $L\not\simeq 0$, M in $\mathsf{D}^{\mathsf{f}}_{+}(S)$, and N in $\mathsf{D}^{\mathsf{f}}_{-}(S)$ the following hold:

$$L\otimes^{\mathsf{L}}_R M\in\mathsf{D}^{\mathsf{f}}_{+}(S) \quad\text{and}\quad L\otimes^{\mathsf{L}}_R M\not\simeq 0 \quad\text{when}\quad M\not\simeq 0 \tag{2.1.1}$$

$$\mathsf{RHom}_R(L,N)\in\mathsf{D}^{\mathsf{f}}_{-}(S) \quad\text{and}\quad \mathsf{RHom}_R(L,N)\not\simeq 0 \quad\text{when}\quad N\not\simeq 0 \tag{2.1.2}$$

2.2. Given a finite subset $\boldsymbol{x}$ of a commutative ring S, let $K[\boldsymbol{x};S]$ denote the Koszul complex on $\boldsymbol{x}$. For each complex M of S-module, set $K[\boldsymbol{x};M]=K[\boldsymbol{x};S]\otimes_S M$. The classical isomorphism $K[\boldsymbol{x};S]\cong\Sigma^{-e}\operatorname{Hom}_S(K[\boldsymbol{x};S],S)$, where $e=\operatorname{card}\boldsymbol{x}$, yields an isomorphism $K[\boldsymbol{x};M]\cong\Sigma^{-e}\operatorname{Hom}_S(K[\boldsymbol{x};S],M)$ of complexes of S-modules.

Let $(S,\mathfrak{n},k)$ be a *local ring*; here this means that S is a commutative Noetherian ring with unique maximal ideal $\mathfrak{n}$, and $l=S/\mathfrak{n}$ is its residue field. When $\boldsymbol{x}$ is a minimal generating set for $\mathfrak{n}$, the complex $K[\boldsymbol{x};M]$ is independent of the choice of $\boldsymbol{x}$, up to isomorphism, so we write K^M in place of $K[\boldsymbol{x};M]$.

For the rest of the paper, we fix a *local homomorphism* $\varphi\colon(R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$; that is, a homomorphism of rings $\varphi\colon R\to S$, satisfying $\varphi(\mathfrak{m})\subseteq\mathfrak{n}$. Set

$$\operatorname{edim}\varphi=\operatorname{edim}(S/\mathfrak{m}S)\,.$$

2.3. Let $\boldsymbol{y}$ in S be a *minimal set of generators of* $\mathfrak{n}$ *modulo* $\mathfrak{m}S$, by which we mean that it contains $\operatorname{edim}\varphi$ elements and its image in $S/\mathfrak{m}S$ generates the ideal $\mathfrak{n}/\mathfrak{m}S$.

For M in $\mathsf{D}^{\mathsf{f}}_{+}(S)$ each S-module $\operatorname{Tor}^R_n(k,K[\boldsymbol{y};M])$ is finite, and is equal to zero for $n\ll 0$; see, (2.1.1). It is annihilated by $\mathfrak{n}$, see [2, 1.5.6], so it is an l-vector space of finite rank. By definition, the nth *Betti number of M over φ* is the integer

$$\beta^{\varphi}_n(M)=\operatorname{rank}_l\operatorname{Tor}^R_n(k,K[\boldsymbol{y};M])\geq 0\,,$$

and the *Poincaré series* of M over φ is the formal Laurent series

$$P^{\varphi}_M(t)=\sum_{n\in\mathbb{Z}}\beta^{\varphi}_n(M)t^n\in\mathbb{Z}[\![t]\!]\,.$$

In case $\varphi=\mathrm{id}^R$, one gets the usual Betti numbers and Poincaré series over R.

When $\boldsymbol{x}$ is a set of generators of $\mathfrak{n}$ containing q elements, the proof of [2, 4.3.1] (where it is assumed that M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(S)$) applies *verbatim* to give an equality

$$P^{\varphi}_M(t)(1+t)^{q-\operatorname{edim}\varphi}=\sum_{n\in\mathbb{Z}}\operatorname{rank}_l\operatorname{Tor}^R_n(k,K[\boldsymbol{x};M])t^n\,. \tag{2.3.1}$$

Choosing $\boldsymbol{x}$ minimal one sees that $P^{\varphi}_M(t)$, and thus $\beta^{\varphi}_n(M)$, does not depend on $\boldsymbol{y}$.

2.4. For M in $\mathsf{D}^{\mathsf{f}}_{+}(S)$, the *curvature* and the *complexity* of M over φ are the numbers

$$\operatorname{curv}_{\varphi}M=\limsup_{n}\sqrt[n]{\beta^{\varphi}_n(M)} \tag{2.4.1}$$

$$\operatorname{cx}_{\varphi}M=\inf\left\{d\in\mathbb{N}\;\middle|\;\begin{array}{c}\text{there exists } c\in\mathbb{R}\text{ such that}\\ \beta^{\varphi}_n(M)\leq cn^{d-1}\text{ for all } n\gg 0\end{array}\right\} \tag{2.4.2}$$

In case $\varphi=\mathrm{id}^R$, we write $\operatorname{curv}_R M$ and $\operatorname{cx}_R M$, respectively.

When M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(S)$ the following inequalities hold, see [2, 7.1.3(5)]:

$$\operatorname{curv}_\varphi M \leq \operatorname{curv}_R k < \infty \quad \text{and} \quad \operatorname{cx}_\varphi M \leq \operatorname{cx}_R k\,. \tag{2.4.3}$$

If, in addition, the ring $S/\varphi(\mathfrak{m})S$ is artinian, [2, 7.2.3] yields

$$\operatorname{curv}_\varphi M = \limsup_n \sqrt[n]{\ell_R \operatorname{Tor}_n^R(k,M)}\,. \tag{2.4.4}$$

2.5. For N in $\mathsf{D}^{\mathsf{f}}_{-}(S)$, the nth *Bass number* $\mu^n_\varphi(N)$ *of* N *over* φ is the integer

$$\mu^n_\varphi(N) = \operatorname{rank}_l \operatorname{Ext}_R^{n-\operatorname{edim}\varphi}(k, K[\boldsymbol{y};N]) \geq 0\,,$$

with $\boldsymbol{y}$ as in 2.3, and the *Bass series* of M over φ is the formal Laurent series

$$I^M_\varphi(t) = \sum_{n\in\mathbb{Z}} \mu^n_\varphi(M)t^n \in \mathbb{Z}[\![t]\!]\,.$$

With $\boldsymbol{x}$ as in 2.3, the proof of [2, 4.3.1] applies *verbatim* to give an equality

$$I^M_\varphi(t)(1+t)^{q-\operatorname{edim}\varphi} = \sum_{n\in\mathbb{Z}} \operatorname{rank}_l \operatorname{Ext}^n_R(k, K[\boldsymbol{x};M])t^n\,. \tag{2.5.1}$$

As above, this implies that $I^M_\varphi(t)$ and $\mu^n_\varphi(N)$ are, indeed, invariants of M.

The obvious analogs of (2.4.1) and (2.4.2) define new asymptotic invariants of N over φ: its *injective curvature* $\operatorname{inj\,curv}_\varphi N$ and its *injective complexity* $\operatorname{inj\,cx}_\varphi N$. Furthermore, the analog of (2.4.4) holds, again by [2, 7.2.3].

3. Duality and compositions

In this section we study the behavior of complexities and curvatures under formation of Matlis duals and compositions of local homomorphisms. For expository reasons, we extend the notation for complexity and curvature.

3.1. Let $a(t) = \sum_{n=i}^\infty a_n t^n$ be a formal Laurent series with a_n real and non-negative.

We set $\operatorname{curv} a(t) = \limsup_n \sqrt[n]{a_n}$ and let $\operatorname{cx} a(t)$ denote the least natural number d such that, for some $c \in \mathbb{R}$ one has $a_n \leq cn^{d-1}$ for all $n \gg 0$.

Let $b(t) = \sum_{n=i}^\infty b_n t^n$ be a Laurent series with b_n real and non-negative.

We write $a(t) \preccurlyeq b(t)$ when $a_n \leq b_n$ holds for each $n \in \mathbb{Z}$; clearly, one then has

$$\operatorname{curv} a(t) \leq \operatorname{curv} b(t) \quad \text{and} \quad \operatorname{cx} a(t) \leq \operatorname{cx} b(t)\,. \tag{3.1.1}$$

The product $a(t)b(t)$ satisfies the following (in)equalities:

$$\operatorname{curv}(a(t)b(t)) = \max\{\operatorname{curv} a(t), \operatorname{curv} b(t)\} \tag{3.1.2}$$

$$\max\{\operatorname{cx} a(t), \operatorname{cx} b(t)\} \leq \operatorname{cx}(a(t)b(t)) \leq \operatorname{cx} a(t) + \operatorname{cx} b(t) \tag{3.1.3}$$

Indeed, $\operatorname{curv} a(t)$ is the reciprocal of the radius of convergence of $a(t)$, hence $\operatorname{curv}(a(t)b(t)) \leq \max\{\operatorname{curv} a(t), \operatorname{curv} b(t)\}$. For the converse, we may assume $a_n \neq 0$ for some n; then $a(t)b(t) \succcurlyeq a_n t^n b(t)$ holds, so (3.1.1) yields the inequality below:

$$\operatorname{curv}(a(t)b(t)) \geq \operatorname{curv}(a_n t^n b(t)) = \operatorname{curv} b(t)\,.$$

By symmetry, we also have $\operatorname{curv}(a(t)b(t)) \geq \operatorname{curv} a(t)$, as desired.

The estimates for $\operatorname{cx}(a(t)b(t))$ are equally easy to verify.

Proposition 3.2. *If E is an injective hull of l over S and M a complex in $\mathsf{D}^{\mathsf{f}}_{-}(S)$, then the complex $N = \operatorname{Hom}_S(K^M, E)$ is in $\mathsf{D}^{\mathsf{f}}_{+}(S)$ and the following equalities hold:*

$$P^{\varphi}_N(t) = I^M_{\varphi}(t)(1+t)^{\operatorname{edim} S}\,,$$
$$\operatorname{curv}_{\varphi} N = \operatorname{inj}\operatorname{curv}_{\varphi} M \quad \textit{and} \quad \operatorname{cx}_{\varphi} N = \operatorname{inj}\operatorname{cx}_{\varphi} M\,.$$

Proof. We first show that $\ell_S(\mathrm{H}_n(K^M))$ is finite for each n, and is zero for $n \gg 0$.

Set $d = \operatorname{edim} S$. The filtration $(K^S_{\leqslant p} \otimes_S M)_p$ yields a spectral sequence with

$$E^2_{p,q} = \mathrm{H}_p\left(K^{\mathrm{H}_q(M)}\right) \quad \text{and} \quad d^r_{p,q}\colon E^r_{p,q} \to E^r_{p-r,q+r-1}\,.$$

The definition of K^S yields $E^2_{p,q} = 0$ for $p \le -1$ and for $p \ge (d+1)$. It follows that $E^r_{p,q} = E^{r+1}_{p,q}$ holds for $r \ge d$, so the spectral sequence converges to $\mathrm{H}_{p+q}(K^M)$.

The hypothesis $\mathrm{H}_q(M) = 0$ for $q \gg 0$ yields $E^2_{p,q} = 0$ for $q \gg 0$, which implies $\mathrm{H}_n(K^M) = 0$ for $n \gg 0$, due to the convergence of the sequence. Moreover, $\mathrm{H}_p(K^{\mathrm{H}_q(M)})$ is Noetherian along with $\mathrm{H}_q(M)$, and is annihilated by $\mathfrak{n}$, so each $E^2_{p,q}$ has finite length; the convergence of the sequence implies that so does $\mathrm{H}_n(K^M)$.

By the injectivity of E, for every $n \in \mathbb{Z}$ there is an isomorphism of S-modules

$$\mathrm{H}_n(N) = \mathrm{H}_n \operatorname{Hom}_S(K^M, E) \cong \operatorname{Hom}_S(\mathrm{H}_{-n}(M), E)\,,$$

which shows that $\mathrm{H}_n(N)$ is finite for each n and is zero for $n \ll 0$.

Set $e = \operatorname{edim} \varphi$, and let $\boldsymbol{y}$ be a minimal generating set of $\mathfrak{n}$ modulo $\mathfrak{m}S$. From the definitions, 2.2, and adjunction we get isomorphisms of complexes of S-modules

$$\begin{aligned}
K[\boldsymbol{y}; N] &= K[\boldsymbol{y}; S] \otimes_S \operatorname{Hom}_S(K^M, E)\\
&\cong \operatorname{Hom}_S\left(\Sigma^{-e} K[\boldsymbol{y}; S], \operatorname{Hom}_S(K^M, E)\right)\\
&\cong \operatorname{Hom}_S\left((\Sigma^{-e} K[\boldsymbol{y}; S] \otimes_S K^M), E\right)\\
&= \operatorname{Hom}_S(\Sigma^{-e} K[\boldsymbol{y}; K^M], E)
\end{aligned}$$

They explain the first one in the following string of isomorphisms in $\mathsf{D}(S)$:

$$\begin{aligned}
k \otimes^{\mathsf{L}}_R K[\boldsymbol{y}; N] &\simeq k \otimes^{\mathsf{L}}_R \left(\Sigma^{-e} \operatorname{Hom}_S(K[\boldsymbol{y}; K^M], E)\right)\\
&\simeq \operatorname{Hom}_S(\Sigma^{-e} \mathsf{R}\operatorname{Hom}_R(k, K[\boldsymbol{y}; K^M]), E)\,.
\end{aligned}$$

Th second one holds because k has a resolution by finite free R-modules, while M is in $\mathsf{D}^{\mathsf{f}}_{-}(S)$ and E is injective. Since E is an injective envelope of l, we obtain the first and the third isomorphisms of l-vector spaces in the string

$$\begin{aligned}
\operatorname{Tor}^R_n(k, K[\boldsymbol{y}; N]) &\cong \operatorname{Hom}_S(\operatorname{Ext}^{n-e}_R(k, K[\boldsymbol{y}; K^M]), E)\\
&\cong \operatorname{Hom}_l(\operatorname{Ext}^{n-e}_R(k, K[\boldsymbol{y}; K^M]), \operatorname{Hom}_S(l, E))\\
&\cong \operatorname{Hom}_l(\operatorname{Ext}^{n-e}_R(k, K[\boldsymbol{y}; K^M]), l)\,.
\end{aligned}$$

The second isomorphism holds because $\mathfrak{n}$ annihilates $\operatorname{Ext}^*_R(k, K[\boldsymbol{y}; K^M])$. Therefore, $\beta^{\varphi}_n(N) = \mu^n_{\varphi}(K^M)$ holds for each n. From this and (2.5.1), we get

$$P^{\varphi}_N(t) = I^{K^M}_{\varphi}(t) = I^M_{\varphi}(t) \cdot (1+t)^d\,.$$

The formulas for curvature and complexity follow, due to (3.1.2) and (3.1.3). □

For $M = S$, the following result reduces to [2, 9.1.1(1)].

Proposition 3.3. *Let $\rho\colon R' \to R$ and $\varphi\colon R \to S$ be local homomorphisms. For each $L \in \mathsf{D}^{\mathsf{f}}_{+}(R)$ and $M \in \mathsf{D}^{\mathsf{f}}_{+}(S)$ there are inequalities:*

$$\begin{aligned}\operatorname{curv}_{\varphi\circ\rho}(L \otimes^{\mathsf{L}}_R M) &\le \max\{\operatorname{curv}_\rho L, \operatorname{curv}_\varphi M\}\,,\\ \operatorname{cx}_{\varphi\circ\rho}(L \otimes^{\mathsf{L}}_R M) &\le \operatorname{cx}_\rho L + \operatorname{cx}_\varphi M\,.\end{aligned}$$

Proof. Let $\mathfrak{m}'$, $\mathfrak{m}$, and $\mathfrak{n}$ denote the maximal ideals of R', R, and S, respectively. Let $\boldsymbol{y}'$ be a minimal generating set of $\mathfrak{m}$ modulo $\mathfrak{m}'R$, let $\boldsymbol{y}$ be one of $\mathfrak{n}$ modulo $\mathfrak{m}S$ and set $\boldsymbol{z} = \varphi(\boldsymbol{y}') \sqcup \boldsymbol{y}$. The isomorphism $\mathfrak{n}/\mathfrak{m}S \cong (\mathfrak{n}/\mathfrak{m}'S)/(\mathfrak{m}S/\mathfrak{m}'S)$ implies that $\boldsymbol{z}$ generates $\mathfrak{n}/\mathfrak{m}'S$. Setting $d = \operatorname{edim}\rho + \operatorname{edim}\varphi - \operatorname{edim}(\varphi\circ\rho)$, and noticing that $L \otimes^{\mathsf{L}}_R M$ is in $\mathsf{D}^{\mathsf{f}}_{+}(T)$ by (2.1.1), we may apply (2.3.1) to obtain

$$P^{\varphi\rho}_{L\otimes^{\mathsf{L}}_R M}(t)(1+t)^d = \sum_{n\in\mathbb{Z}} \operatorname{rank}_l \operatorname{Tor}^R_n(k, K[\boldsymbol{z}; L\otimes^{\mathsf{L}}_R M])t^n\,. \tag{3.3.1}$$

In the derived category of S, the isomorphism

$$K[\boldsymbol{z}; L\otimes^{\mathsf{L}}_R M] \simeq K[\boldsymbol{y}'; L] \otimes^{\mathsf{L}}_R K[\boldsymbol{y}; M]\,,$$

combined with the associativity formula for derived tensor products yields

$$\big(k' \otimes^{\mathsf{L}}_{R'} K[\boldsymbol{y}'; L]\big) \otimes^{\mathsf{L}}_R K[\boldsymbol{y}; M] \simeq k' \otimes^{\mathsf{L}}_{R'} K[\boldsymbol{z}; L\otimes^{\mathsf{L}}_R M]\,.$$

This isomorphism gives rise to a standard spectral sequence with

$$E^2_{pq} = \operatorname{Tor}^R_p\big(\operatorname{Tor}^{R'}_q(k', K[\boldsymbol{y}'; L]), K[\boldsymbol{y}; M]\big) \implies \operatorname{Tor}^{R'}_{p+q}\big(k', K[\boldsymbol{z}; L\otimes^{\mathsf{L}}_R M]\big)\,.$$

The R-module $\operatorname{Tor}^{R'}(K[\boldsymbol{y}'; L], k')$ is annihilated by $\mathfrak{m}$, so one has

$$\operatorname{Tor}^R_p\big(\operatorname{Tor}^{R'}_q(k', K[\boldsymbol{y}'; L]), K[\boldsymbol{y}; M]\big) \cong \operatorname{Tor}^{R'}_q(k', K[\boldsymbol{y}'; L]) \otimes_{k'} \operatorname{Tor}^R_p(k, K[\boldsymbol{y}; M])\,.$$

The preceding isomorphism and the convergence of the spectral sequence yield

$$\sum_{n\in\mathbb{Z}} \operatorname{rank}_l \operatorname{Tor}^R_n(k, K[\boldsymbol{z}; L\otimes^{\mathsf{L}}_R M])t^n \preccurlyeq P^\rho_L(t)\cdot P^\varphi_M(t)\,. \tag{3.3.2}$$

Combining formulas (3.3.1) and (3.3.2), we get a coefficientwise inequality

$$P^{\varphi\circ\rho}_{L\otimes^{\mathsf{L}}_R M}(t)\cdot(1+t)^d \preccurlyeq P^\rho_L(t)\cdot P^\varphi_M(t)$$

which, by (3.1.1), implies the inequality in the following string:

$$\begin{aligned}\operatorname{curv}_{\varphi\circ\rho}(L\otimes^{\mathsf{L}}_R M) &= \operatorname{curv} P^{\varphi\circ\rho}_{L\otimes^{\mathsf{L}}_R M}(t)\\ &= \operatorname{curv}\big(P^{\varphi\circ\rho}_{L\otimes^{\mathsf{L}}_R M}(t)\cdot(1+t)^d\big)\\ &\le \operatorname{curv}\big(P^\rho_L(t)\cdot P^\varphi_M(t)\big)\\ &= \max\{\operatorname{curv} P^\rho_L(t), \operatorname{curv} P^\varphi_M(t)\}\\ &= \max\{\operatorname{curv}_\rho L, \operatorname{curv}_\varphi M\}\,.\end{aligned}$$

The equalities at both ends hold by definition, the other two by (3.1.2).

A similar argument, using (3.1.3), yields $\operatorname{cx}_{\varphi\circ\rho}(L\otimes^{\mathsf{L}}_R M) \le \operatorname{cx}_\rho L + \operatorname{cx}_\varphi M$. □

4. Homotopical Loewy length

In this section $(S,\mathfrak{n},l)$ is a local ring and M a complex of S-modules.

We introduce two notions that play a critical, if behind-the-scenes, role in the proof of our main results. The Loewy length of the complex M is the number

$$\ell\ell_S M = \inf\{i \in \mathbb{N} \mid \mathfrak{n}^i M = 0\}\,.$$

The *homotopical Loewy length* of M is defined in [2] to be the number

$$\ell\ell_{\mathsf{D}(S)} M = \inf\{\ell\ell_S V \mid M \simeq V \text{ in } \mathsf{D}(S)\}\,.$$

The proof of the next result is extracted from that of [2, 6.2.2], which provides a more precise upper bound for the homotopical Loewy length of K^S.

Proposition 4.1. *Every complex M over a local ring $(S,\mathfrak{n},l)$ satisfies*

$$\ell\ell_{\mathsf{D}(S)} K^M \le \ell\ell_{\mathsf{D}(S)} K^S < \infty\,.$$

Proof. Set $d = \operatorname{edim} S$. For all integers $i \gg 0$ and all $n \in \mathbb{Z}$, the subcomplex

$$J^i = \quad 0 \to \mathfrak{n}^{i-d} K^S_d \to \mathfrak{n}^{i-d+1} K^S_{d-1} \to \cdots \to \mathfrak{n}^{i-1} K^S_1 \to \mathfrak{n}^i K^S_0 \to 0$$

of K^S satisfies $\mathrm{H}_n(J^i) = 0$, by a well-known result of Serre; see [1, 4.1.6(3)]. For such an i, the canonical map $K^S \to K^S/J^i$ is a quasi-isomorphism, so it represents an isomorphism in $\mathsf{D}(S)$. Now $\mathfrak{n}^i(K^S/J^i) = 0$ implies $\ell\ell_{\mathsf{D}(S)} K^S \le i < \infty$.

Set $c = \ell\ell_{\mathsf{D}(S)} K^S$. Let $\varkappa\colon K^S \xrightarrow{\simeq} V$ be an isomorphism in $\mathsf{D}(S)$, with $\mathfrak{n}^c V = 0$, and let $\varepsilon\colon F \xrightarrow{\simeq} M$ be a semifree resolution. The quasi-isomorphisms of complexes

$$K^M = K^S \otimes_S M \xleftarrow[\simeq]{F\otimes_S \varepsilon} K^S \otimes_S F \xrightarrow[\simeq]{\varkappa \otimes_S F} V \otimes_S F$$

represent an isomorphism $K^M \simeq V \otimes_S F$ in $\mathsf{D}(S)$. It implies $\ell\ell_{\mathsf{D}(S)} K^M \le c$, since

$$\mathfrak{n}^c(V \otimes_S F) = (\mathfrak{n}^c V) \otimes_S F = 0\,. \qquad \square$$

Recall that $V \in \mathsf{D}(S)$ is *formal* if there is an isomorphism $V \simeq \mathrm{H}(V)$ in $\mathsf{D}(S)$.

Remark 4.2. If $\mathrm{H}(V)$ is projective, then V is formal.

Indeed, choosing for each $n \in \mathbb{Z}$ a splitting $\sigma_n\colon \mathrm{H}_n(V) \to \mathrm{Z}_n(V)$ of the canonical surjection $\mathrm{Z}_n(V) \to \mathrm{H}_n(V)$, and composing σ_n with the inclusion $\mathrm{Z}_n(V) \to V_n$, one gets a quasi-isomorphism $\mathrm{H}(V) \to V$, whence an isomorphism $\mathrm{H}(V) \cong V$ in $\mathsf{D}(S)$.

Proposition 4.3. *Let $(S,\mathfrak{n},l)$ be a local ring and set $c = \ell\ell_{\mathsf{D}(S)} K^S$.*

If $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ is a local homomorphism with $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}^c$, then for every complex M of S-modules the following assertions hold.

(1) *The complex K^M is formal in $\mathsf{D}(R)$.*

(2) *For each L in $\mathsf{D}(R)$ there are isomorphisms of graded l-vector spaces*

$$\operatorname{Tor}^R_*(L, K^M) \cong \operatorname{Tor}^R_*(L,k) \otimes_k \mathrm{H}_*(K^M)\,.$$

(3) *If M is in $\mathsf{D}^{\mathsf{f}}_+(S)$ and $M \not\simeq 0$, then there are inequalities*

$$\operatorname{curv}_\varphi M \ge \operatorname{curv}_R k \quad \text{and} \quad \operatorname{cx}_\varphi M \ge \operatorname{cx}_R k\,.$$

(4) *If M is in $\mathsf{D}^{\mathsf{f}}_-(S)$ and $M \not\simeq 0$, then there are inequalities*

$$\operatorname{inj\,curv}_\varphi M \ge \operatorname{curv}_R k \quad \text{and} \quad \operatorname{inj\,cx}_\varphi M \ge \operatorname{cx}_R k\,.$$

(5) *If M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(S)$ and $M \not\simeq 0$, then equalities hold in* (3) and (4).

Proof. (1). Proposition 4.1 yields in $\mathsf{D}(S)$ an isomorphism $K^M \simeq V$, with $\mathfrak{n}^c V = 0$. This implies $\mathfrak{m} \cdot V = 0$, so R acts on V through k. Since k is a field, V is formal in $\mathsf{D}(k)$, see Remark 4.2, and hence also in $\mathsf{D}(R)$.

(2). From (1) we get the first one of the following isomorphisms in $\mathsf{D}(R)$:

$$L \otimes^{\mathsf{L}}_R K^M \simeq L \otimes^{\mathsf{L}}_R \mathrm{H}(K^M) \simeq (L \otimes^{\mathsf{L}}_R k) \otimes_k \mathrm{H}(K^M)\,.$$

The second one holds because $\mathfrak{m} \cdot \mathrm{H}(K^M) = 0$. Now pass to homology and use the Künneth isomorphism.

(3) and (5). Set $e = \operatorname{edim} S - \operatorname{edim} \varphi$ and $h(t) = \sum_{n \in \mathbb{Z}} \operatorname{rank}_l \mathrm{H}_n(K^M) t^n$.

The isomorphism in (2), applied with $L = k$, gives $P^{\varphi}_M(t)(1+t)^e = P^R_k(t) \cdot h(t)$. This explains the middle equality in the following display, where the first and last ones hold by definition, while the remaining two come from (3.1.2):

$$\begin{aligned} \operatorname{curv}_{\varphi} M &= \operatorname{curv} P^{\varphi}_M(t) \\ &= \operatorname{curv} \big(P^{\varphi}_M(t) \cdot (1+t)^e\big) \\ &= \operatorname{curv} \big(P^R_k(t) \cdot h(t)\big) \\ &= \max\{\operatorname{curv} P^R_k(t), \operatorname{curv} h(t)\} \\ &= \max\{\operatorname{curv}_R k, \operatorname{curv} h(t)\}\,. \end{aligned}$$

It remains to note that $\operatorname{curv} h(t) \geq 0$ holds, with equality when M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(S)$.

A similar argument, using (3.1.3), yields the assertions concerning $\operatorname{cx}_{\varphi} M$.

(4) and (5). This follows from (3) and (5), due to Proposition 3.2 □

5. Contracting endomorphisms

An endomorphism $\phi \colon R \to R$ of a local ring $(R, \mathfrak{m}, k)$ is said to be *contracting* if for every r in $\mathfrak{m}$ the sequence $(\phi^i(r))_{i \geq 1}$ converges to zero in the $\mathfrak{m}$-adic topology of R. Necessary conditions and sufficient conditions for the existence of such endomorphisms are discussed in Remark 5.9 and Example 5.10, respectively.

Now we present the main result of the paper.

Theorem 5.1. *Let $\phi \colon R \to R$ a contracting endomorphism of a local ring $(R, \mathfrak{m}, k)$. For each $i \geq 1$ and each complex M in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ with $M \not\simeq 0$ there are equalities*

$$\operatorname{curv}_{\phi^i} M = \operatorname{curv}_R k = \operatorname{inj\,curv}_{\phi^i} M\,.$$

Some special cases of the theorem are known from earlier work.

Remark 5.2. Assume that M is in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ and $M \not\simeq 0$.

The equalities in the theorem hold for all $i \gg 1$ by [2, 12.1.3].

When M is a bounded complexes of free R-modules, one gets $\operatorname{curv}_{\phi^i} M = \operatorname{curv}_R k$ for all $i \geq 1$ by [3, 5.10] and [2, 12.1.5]. When, in addition, the ring R is Gorenstein, [3, 5.11] and [2, 12.1.5] yield $\operatorname{inj\,curv}_{\phi^i} M = \operatorname{curv}_R k$ for all $i \geq 1$.

Extending the notation for modules, we write ${}^{\phi}M$ for the complex with the same underlying graded abelian group as M and R-action given by $r \cdot m = \phi(r)m$.

Proof of Theorem 5.1. It suffices to treat the case $i = 1$, for ϕ^i is contracting for each $i \geq 1$. Moreover, by Proposition 3.2, it suffices to prove $\operatorname{curv}_{\phi} M = \operatorname{curv}_R k$.

Set $M^{(1)} = M$ and for each integer $n \geq 2$ define, inductively, a complex

$$M^{(n)} = M^{(n-1)} \otimes^{\mathsf{L}}_R {}^{\phi}M$$

in $\mathsf{D}(R)$, where the action of R on $M^{(n)}$ is obtained by applying 2.1 to $\phi\colon R\to R$, with $L=M^{(n-1)}$ and M. Thus, it is induced by the action on the additive group of ${}^{\phi}M$, coming from the original action of R on the additive group of M.

We claim that for $n\geq 1$ the following statements hold:

(1_n) $M^{(n)}$ is in $\mathsf{D}^{\mathsf{f}}_{+}(R)$ and $\mathrm{H}(M^{(n)})\neq 0$.

(2_n) $\operatorname{curv}_{\phi^n} M^{(n)}\leq \operatorname{curv}_{\phi} M$.

Indeed, both assertions are tautological for $n=1$, so we may assume that they hold for some $n\geq 1$. Now (2.1.1) and the induction hypothesis give (1_{n+1}). To obtain (2_{n+1}) we use the following relations, which come from Proposition 3.3 applied with $R'=R=S$, $\rho=\phi^n$, and $\varphi=\phi$, and from the induction hypothesis

$$\begin{aligned}\operatorname{curv}_{\phi^{n+1}} M^{(n+1)} &= \operatorname{curv}_{\phi\circ\phi^n}\left(M^{(n)}\otimes^{\mathsf{L}}_R {}^{\phi}M\right)\\ &\leq \max\{\operatorname{curv}_{\phi^n} M^{(n)},\operatorname{curv}_{\phi} M\}\\ &\leq \max\{\operatorname{curv}_{\phi} M,\operatorname{curv}_{\phi} M\}\\ &= \operatorname{curv}_{\phi} M\,.\end{aligned}$$

Set $c=\ell\ell_{\mathsf{D}(S)}K^S$. As ϕ is contracting, we have $\phi^s(\mathfrak{m})\subseteq\mathfrak{m}^c$ for some integer s. Applying Proposition 4.3(3), assertion (2_s) above, and (2.4.3) we now get

$$\operatorname{curv}_R k\leq \operatorname{curv}_{\phi^s} M^{(s)}\leq \operatorname{curv}_{\phi} M\leq \operatorname{curv}_R k\,. \qquad \square$$

The notation and hypotheses of the theorem are kept in force in its corollaries.

Part (1) of the first corollary contains Theorem 1.1, announced in the introduction.

Corollary 5.3. *For each positive integer i the following hold.*

(1) *If ${}^{\phi^i}M$ is isomorphic in $\mathsf{D}(R)$ to a bounded complex of flat R-modules, or to a bounded complex of injective R-modules, then R is regular.*

(2) *If $\operatorname{curv}_{\phi^i} M\leq 1$ or $\operatorname{inj\,curv}_{\phi^i} M\leq 1$ holds, then R is complete intersection.*

Remark 5.4. Part (1) of the corollary contains Rodicio's generalization of Kunz's Theorem: When R is of characteristic $p>0$ and ϕ is the Frobenius map, if ${}^{\phi^i}R$ has finite flat dimension for some i then R is regular; see [6, Thm. 2].

Proof of Corollary 5.3. (1) The hypotheses on ${}^{\phi^i}M$ imply $\beta^{\phi^i}_n(M)=0$ or $\mu^n_{\phi^i}(M)=0$ for all $n\gg 0$, whence $\operatorname{curv}_{\phi^i} M=0$ or $\operatorname{inj\,curv}_{\phi^i} M=0$. The theorem then yields $\operatorname{curv}_R k=0$, so R is regular by the Auslander-Buchsbaum-Serre Theorem.

(2) The theorem gives $\operatorname{curv}_R k\leq 1$, so R is complete intersection by [1, 8.2.2]. $\square$

The corollary *characterizes* regularity and complete intersection, since it is known that the converses of both (1) and (2) hold. This follows immediately from the precise information available on the asymptotic behavior of Betti numbers and Bass numbers over contracting endomorphisms of complete intersections.

Remark 5.5. When R is complete intersection, [2, 5.3.2] yields for each M in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ polynomials $b^M_{\pm}(t)\in\mathbb{Q}[t]$ with the same leading term and of degree at most $\operatorname{codim} R-1$, such that Betti numbers $\beta^{\phi}_n(M)$ satisfy the equalities

$$\beta^{\phi}_n(M)=\begin{cases} b^M_+(n) & \text{for all even } n\gg 0\,,\\ b^M_-(n) & \text{for all odd } n\gg 0\,.\end{cases}$$

Furthermore, the Bass numbers $\mu^n_{\phi}(M)$ have a similar property.

In [2], a complex M in $\mathsf{D}^{\mathsf{f}}_{\mathsf{b}}(R)$ is said to be *extremal* over ϕ if it satisfies

$$\operatorname{curv}_\phi M = \operatorname{curv}_R k \quad \text{and} \quad \operatorname{cx}_\phi M = \operatorname{cx}_R k\,.$$

The obvious substitutions yield a definition of *injective extremality*.

Part (b) of the next corollary answers, in the positive, Question [2, 12.2.2].

Corollary 5.6. *Under any one of the following conditions, M is extremal over ϕ^i for $i \geq 1$:*

(a) *The ring R is not complete intersection.*
(b) *The ring R has positive characteristic and ϕ is the Frobenius endomorphism.*

Proof. Theorem 5.1 shows that we need only compare complexities.

Condition (a) implies $\operatorname{curv}_R k > 1$ and $\operatorname{cx}_R k = \infty$ by [1, 8.2.2], so from Theorem 5.1 we obtain $\operatorname{curv}_{\phi^i} M > 1 < \operatorname{inj\,curv}_{\phi^i} M$, whence $\operatorname{cx}_{\phi^i} M = \infty = \operatorname{inj\,cx}_{\phi^i} M$.

Under condition (b), the equalities of complexities are proved in [2, 12.2.4]. □

The restriction in condition (a) is essential:

Remark 5.7. When R is complete intersection, M is extremal and injectively extremal over ϕ^i for $i \gg 0$, by [2, 12.1.3], but not in general, see [2, 12.1.6].

Theorem 1.2 from the introduction is contained in the next corollary. It follows from Theorem 5.1, formula (2.4.4), and its analog for injective complexity, see 2.5.

Corollary 5.8. *When $R/\phi(\mathfrak{m})R$ is artinian the following equalities hold for $i \geq 1$:*

$$\limsup_n \sqrt[n]{\ell_R \operatorname{Tor}^R_n(k, {}^{\phi^i}\!M)} = \operatorname{curv}_R k = \limsup_n \sqrt[n]{\ell_R \operatorname{Ext}^n_R(k, {}^{\phi^i}\!M)}\,. \qquad \square$$

In order to apply our results to a given ring R, one needs to know that it admits *some* contracting endomorphism. Mohan Kumar and Hamid Rahmati have noticed that such a ring has to be *equicharacteristic*; that is, to satisfy $\operatorname{char}(k)R = 0$.

Remark 5.9. If R admits a contracting endomorphism, then it is equicharacteristic.

More precisely, if $\phi\colon R \to R$ is a contracting endomorphism, then the set

$$k_0 = \{r \in R \mid \phi(r) = r\}$$

is a subfield of R. Indeed, it is immediately clear that k_0 is a subring of R. For $r \in k_0 \cap \mathfrak{m}$ one has $r \in \bigcap_{j=1}^{\infty} \mathfrak{m}^j = 0$. Thus, each non-zero element r of k_0 has an inverse in R; for every $i \geq 1$ it satisfies $\phi(r^{-1}) = \phi(r)^{-1} = r^{-1}$, so r^{-1} is in k_0.

Conversely, equicharacteristic rings often have contracting endomorphisms:

Examples 5.10. (1) If R is equicharacteristic and $\operatorname{char}(k) = p > 0$, then the Frobenius map $r \mapsto r^p$ is a contracting endomorphism.

(2) If k is an arbitrary field, B is a finitely generated subsemigroup of $\mathbb{N}^n$ for some integer n, and R is the localization of $k[B]$ at the maximal ideal spanned by the positive elements of B, then for every integer $q \geq 2$ the map $B \to B$ given by $b \mapsto qb$, induces a contracting endomorphism $R \to R$.

(3) If the canonical map $\varepsilon\colon R \to k$ admits a right inverse homomorphism of rings $\sigma\colon k \to R$, then $\sigma\varepsilon\colon R \to R$ is a contracting endomorphism.

In particular, every equicharacteristic and complete local ring admits a contracting endomorphism, due to Cohen's Structure Theorem.

6. A proof of a special case for the Frobenius endomorphism

In this section, we give an entirely different proof of a special case of Theorem 1.1 of the Introduction using tight closure methods in the case where ϕ is a power of the Frobenius endomorphism of R. Precisely:

Theorem 6.1. *Let M be a finitely generated module over a Noetherian ring R of positive prime characteristic p that is supported everywhere on* $\operatorname{Spec}(R)$. *Suppose that $\phi = F^e$ is an iteration of the Frobenius endomorphism F of R and that ${}^{\phi}M$ is R-flat. Then R is regular.*

We make use of bimodules in the sequel. If $\phi\colon R \to S$ is a homomorphism and W is a right S-module, then we may define an (R, S)-bimodule structure on W on the abelian group W such that the left R-module structure is given by restriction of scalars, that is ${}^{\phi}W$, and the right S-module structure is the original one. Thus, $rw = w\phi(r)$ for all $r \in R$ and $w \in W$. In the sequel, a *ϕ-bimodule* means an (R, S)-bimodule isomorphic to one obtained as above.

Remark 6.2. Let $\phi\colon R \to S$ and $\psi\colon S \to T$ be ring homomorphisms, let M be a ϕ-bimodule and N a ψ-bimodule. The following assertions are clear:

If M is finitely generated as an S-module, and N is as a T-module, then $M \otimes_S N$ is finitely generated as a T-module.

If M is flat as an R-module and N is flat as an S-module, then $M \otimes_S N$ is flat as an R-module.

Lemma 6.3. *Let $\phi\colon R \to S$ be a local homomorphism, $\widehat{\phi}\colon \widehat{R} \to \widehat{S}$ the induced homomorphism of complete local rings, and M a ϕ-bimodule.*

1. *If M is finitely generated as an S-module and flat as an R-module, then $M \otimes_S \widehat{S}$ has a structure of $\widehat{\phi}$-bimodule that is finitely generated as an $\widehat{S}$-module and flat as an $\widehat{R}$-module.*
2. *If* $\operatorname{Supp}_S M = \operatorname{Spec} S$, *then* $\operatorname{Supp}_{\widehat{S}}(M \otimes_S \widehat{S}) = \operatorname{Spec} \widehat{S}$.

Proof. (1) We have a commutative diagram, with ι_R and ι_S the canonical maps:

(6.3.1)
$$\begin{array}{ccc} \widehat{R} & \xrightarrow{\widehat{\phi}} & \widehat{S} \\ {\scriptstyle \iota_R}\uparrow & & \uparrow{\scriptstyle \iota_S} \\ R & \xrightarrow{\phi} & S \end{array}$$

Set $\phi' = \iota_S \circ \phi$; this is the same as $\widehat{\phi} \circ \iota_R$. Then $M \otimes_S \widehat{S}$ is a right $\widehat{S}$-module, which in turn gets a structure of a $\widehat{\phi}$-bimodule as well as a ϕ'-bimodule. Note that $M \otimes_S \widehat{S}$ is finitely generated as an $\widehat{S}$-module and flat as an R-module, by Remark 6.2, since M is flat over R and $\widehat{S}$ is flat over S.

Now, we claim that $M \otimes_S \widehat{S}$ is flat also as an $\widehat{R}$-module. In order to prove this, it suffice to show that the $\widehat{R}$-module action on $M \otimes_S \widehat{S}$ preserves inclusions of finitely generated $\widehat{R}$-modules. But if there is a counterexample, then there must be a counterexample involving finite length $\widehat{R}$-modules (cf. the Artin-Rees Lemma, the Krull Intersection Theorem, and Remark 6.2 showing $M \otimes_S \widehat{S}$ is finitely generated over $\widehat{S}$). And these finite length $\widehat{R}$-modules and the inclusion map in the counterexample must (and trivially) come from the category of R-modules via the scalar extension ι_R, which contradicts the flatness of $M \otimes_S \widehat{S}$ over R. (This is the local flatness criterion.)

(2) Note that $\operatorname{Supp}_S M$ is the set of prime ideals in S containing $\operatorname{Ann}_S M$. The desired equality holds because $\operatorname{Ann}_{\widehat{S}}(M \otimes_S \widehat{S}) = (\operatorname{Ann}_S M)\widehat{S}$, since M is a finitely generated S-module and $\widehat{S}$ is flat over S. □

Throughout the remainder of this section, R will denote a Noetherian ring of prime characteristic $p > 0$, e a positive integer, and q will denote p^e. In this case, an F^e-bimodule is a (right) R-module M with left R-module structure given by $rm = mr^q$ for $r \in R$ and $m \in M$. We write M_R (respectively, ${}_RM$) for M viewed as a right (respectively, left) R-module.

Now Theorem 6.1 may be restated as follows:

Theorem 6.4. *Let R be a Noetherian commutative ring of prime characteristic p. Assume there exists an F^e-bimodule M with $e \geq 1$ such that* $\operatorname{Supp}(M) = \operatorname{Spec}(R)$, *the right R-module M_R is finitely generated R-module and the left R-module ${}_RM$ is flat. Then R is regular.*

Proof. We first note that the result reduces at once to the local case. Henceforth, we may assume without loss of generality that $(R, \mathfrak{m})$ is local. The proof proceeds in three steps. We first show that if R is Cohen-Macaulay, then R is regular. Next we show that R must be a domain. Finally, we use tight closure theory to prove that R is, in fact Cohen-Macaulay and hence regular; and we achieve this by reducing to the case where R is complete.

As usual, if I is an ideal of R, $I^{[q]}$ denotes the ideal $(f^q : f \in I)R$, which is the expansion of I under $F^e \colon R \to R$.

For every positive integer n, denote $M^{(n)} := M^{\otimes n}$. That is, form $M^{(n+1)}$ recursively as $M \otimes_R M^{(n)}$. Note that for all n, $M^{(n)}$ is naturally an F^{ne}-bimodule that is finitely generated as a right R-module and flat as a left R-module, by Remark 6.2. In fact, $M^{(n)}$ is automatically faithfully flat as a left R-module, since $(R/\mathfrak{m}) \otimes_R M^{(n)} \cong M^{(n)}/(M^{(n)}\mathfrak{m}^{[q^n]})$ as right R-modules, and the latter module is nonzero by Nakayama's lemma.

First, we show that if $(R, \mathfrak{m})$ is Cohen-Macaulay, then R is regular. In this case M is a (possibly big) Cohen-Macaulay left R-module; hence M is a (small) Cohn-Macaulay right R-module. The same is true for $M^{(n)}$. Let $\boldsymbol{x}$ be a full system of parameters for R (hence, an R-regular sequence). There exists n big enough such that $\mathfrak{m}M^{(n)} \subseteq M^{(n)}(\boldsymbol{x})$. (Note that $\mathfrak{m}^{[q^n]} \subseteq (\boldsymbol{x})$ for $n \gg 0$.) Replace M by $M^{(n)}$ so that we may assume $\mathfrak{m}M \subseteq M(\boldsymbol{x})$ in the remainder of this part of the argument.

Then by considering the complex $M \otimes K[\boldsymbol{x}; R]$, we see $M/(M(\boldsymbol{x}))$ has finite flat dimension as a left R-module. Moreover, in light of $\mathfrak{m}M \subseteq M(\boldsymbol{x})$, we see that $M/(M(\boldsymbol{x}))$ is a non-zero vector space over $R/\mathfrak{m}$, through its structure as a left R-module. Thus $R/\mathfrak{m}$ has finite flat dimension hence R is regular, as claimed.

Next, we show that $(R, \mathfrak{m})$ must be a domain: Let $a \in R \setminus \{0\}$ and consider the exact sequence

$$0 \to I \to R \xrightarrow{a} R\ ,$$

where $I := (0 :_R a)$. Apply $\otimes_R M$ and use the left flatness of ${}_RM$ to get

$$\{x \in M \mid ax = 0\} = IM = MI^{[q]}.$$

However, we see directly that

$$MI \subseteq \{x \in M \mid ax = 0\}.$$

Thus $MI^{[q]} = MI$, and so $MI^2 = MI$. By Nakayama's lemma, we see that $MI = 0$. Hence $IM = 0$ or $I \otimes_R M = 0$. It follows that $I = 0$ by the faithful flatness of M as a left R-module. Therefore, R is automatically a domain as claimed.

Henceforth, we assume $(R, \mathfrak{m})$ is local (hence a domain), and prove the theorem by induction on $\dim(R)$. Being a domain, when $\dim(R) = 0$, R is a field and hence regular. Thus we may assume $\dim(R) \geq 1$.

Let $F_R^e \colon R \to R$ be the Frobenius endomorphism, which is local. Noting that $\widehat{F_R^e}$ is the Frobenius endomorphism $F_{\widehat{R}}^e$ of $\widehat{R}$, and that R is regular when $\widehat{R}$ is regular, one may assume, by Lemma 6.3, that R is complete local.

Therefore, we further assume $(R, \mathfrak{m})$ is a complete domain. By the result proved at the outset, it suffices to show R is Cohen-Macaulay. In fact, we are going to show R is *weakly F-regular* (that is, $I^* = I$ for all ideals of R). To this end, it is enough to show $I = I^*$ for all $\mathfrak{m}$-primary ideals I of R. By way of contradiction, suppose there exists $x \in R \setminus I$ such that $x \in I^*$ for some $\mathfrak{m}$-primary ideal I. Set $J := (I, x)$. By choosing x to be a socle element, we may assume $\ell(J/I) = 1$.

Observe that, for every $P \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$, the induction hypothesis applies to R_P and M_P, which shows R_P is regular. In other words, R is an isolated singularity. Thus, as R is a complete (hence excellent) domain, the *test ideal* of R, denoted τ, is $\mathfrak{m}$-primary: see [4, Theorem 6.20].

In what follows, to indicate that we are taking the number of generators, or length, or the annihilator, of a given module viewing it as a right module, we include the superscript ^{r} in the notation. For instance, $\nu^r(M)$ is the minimal numbers of generators of M_R. Clearly, we have

$$\nu^r(M) = \ell^r(M/M\mathfrak{m}).$$

As R is a domain, $\dim(R) \geq 1$ and M is torsion-free (as a left and hence a right R-module), we have $M\mathfrak{m} \neq 0$. Thus, Nakayama's implies $M\mathfrak{m} \supsetneq M\mathfrak{m}^2 \supseteq M\mathfrak{m}^{[q]}$. Next, setting

$$\begin{aligned} f &= \ell^r(k \otimes_R M) = \ell^r(M/\mathfrak{m}M) = \ell^r(M/M\mathfrak{m}^{[q]}) \quad \text{and} \\ g &= \nu^r(M) = \ell^r(M/M\mathfrak{m}), \end{aligned}$$

the argument in the above paragraph implies

$$f = \ell^r(M/\mathfrak{m}M) = \ell^r(M/M\mathfrak{m}^{[q]}) > \ell^r(M/M\mathfrak{m}) = \nu^r(M) = g. \tag{6.4.1}$$

(Note the strict inequality in (6.4.1).)

We are going to study $\ell^r(JM^{(n)}/IM^{(n)}) = \ell^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]})$ and get a contradiction. As ${}_RM$ and hence $M^{(n)}$ are flat as left R-modules, for all n we get

$$\ell^r(JM^{(n)}/IM^{(n)}) = \ell^r(k \otimes_R M^{(n)}) = f^n\,. \tag{6.4.2}$$

Also notice that $\nu^r(M^{(n)}) \leq g^n$ for all n. (In fact, $\nu^r(M^{(n)}) = g^n$ for all n.)

Now let us study $\ell^r(JM^{(n)}/IM^{(n)}) = \ell^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]})$ via the uniform annihilating property of τ. As $J^{[q^n]} = (I^{[q^n]}, x^{q^n})$, we see that the minimal numbers of generator of $M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]}$ as a right R-module satisfies

$$\nu^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]}) \leq \nu^r(M^{(n)}) \leq g^n$$

for all n. Moreover, we have

$$J^{[q^n]}\tau \subseteq I^{[q^n]},$$

which implies that $M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]}$, as a right R-module, is killed by τ (which has been observed to be $\mathfrak{m}$-primary) for all n. Therefore, we see

$$\ell^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]}) \leq \nu^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]})\ell(R/\tau) \leq g^n\ell(R/\tau)$$

for all n. Consequently, we get

$$\ell^r(JM^{(n)}/IM^{(n)}) = \ell^r(M^{(n)}J^{[q^n]}/M^{(n)}I^{[q^n]}) \leq g^n\ell(R/\tau) \tag{6.4.3}$$

for all n. Finally, as $g < f$ (see (6.4.1)), we must have

$$\ell^r(JM^{(n)}/IM^{(n)}) \leq g^n\ell(R/\tau) < f^n = \ell^r(JM^{(n)}/IM^{(n)})$$

for all sufficiently large n, which is a contradiction. (In other words, (6.4.2) and (6.4.3) contradict each other.)

Thus R is Cohen-Macaulay and therefore R is regular. □

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