F-PURITY, FROBENIUS SPLITTING, AND TIGHT CLOSURE

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1. Introduction

I became interested in the study of F-purity and F-splitting in the interval 1967–1973 while I was at the University of Minnesota. My colleague Jack Eagon and I did work on the properties of determinantal rings (discussed briefly in §2, Example (13)): see [HE]. This led to work, joint with Joel Roberts [HR1], on proving that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay, and ultimately to a further study of F-purity [HR2]. At the same time I became interested in the local homological conjectures. Irving Kaplansky sent me an early preprint of the joint thesis of Peskine and Szpiro, [PS], which was a great source of inspiration for me. I became interested in a number of splitting questions [Ho2, Ho5], in the technique of reduction to characteristic p, and in the existence of big Cohen-Macaulay modules and algebras [Ho3, Ho4, Ho6, HH4, HH6]. This led in turn to the development of tight closure theory [HH1, HH2, HH3, HH5, HH7] in joint work with Craig Huneke that began in the fall of 1986. I will return to these themes below.

2. Pure and split extensions

Throughout, R is a commutative, associative ring with 1. A homomorphism of R-modules $\alpha : N \to M$ is called *pure* if $W \otimes_R N \to W \otimes_R M$ is an injective map for every R-module W. Since we may take $W = R$, we have, in particular, that $N \to M$ must be injective. If N is a direct summand of M, i.e., if there is a splitting $\beta : M \to N$ such that $\beta \circ \alpha = \mathrm{id}_N$, then $N \to M$ is pure. If M/N is finitely presented, then $N \to M$ is pure if and only if N is a direct summand of M. Thus, if R is Noetherian, $N \to M$ is pure if and only if it is a direct limit of split extensions $N \to M_0$, since M is the directed union of its submodules finitely generated over N (this is true even when R is not Noetherian, although the maps $M_0 \to M$ need not be injective in that case). For more detail on purity, see [HH6], pp. 48–50.

A ring extension $R \to S$ is called *split* (respectively, *pure*) if $R \to S$ is split (respectively, pure) as a map of R-modules. When this holds, if $G_\bullet$ is any complex

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of R-modules, the maps $G_\bullet \to S \otimes_R G_\bullet$ are split (respectively, pure), and so are the induced maps of homology or cohomology between the two complexes.

If R has prime characteristic $p > 0$, R is *F-split* (respectively, F-*pure*) if the Frobenius endomorphism $F_R = F : R \to R$ is split (respectively, pure). If either condition holds, R is reduced. When R is reduced, the maps $F : R \to R$, $F(R) \hookrightarrow R$, and $R \hookrightarrow R^{1/p}$ are isomorphic maps.

Examples.

(1) If $R \to S$ is faithfully flat, it is pure.
(2) Splitting and purity for ring homomorphisms are both preserved by composition.
(3) Any map of fields $K \to L$ is split over K, since 1 is part of a free basis for L.
(4) If $R \to S$ is split, say by a map $\alpha : S \to R$, then $R[x_1, \dots, x_n] \to S[x_1, \dots, x_n]$ is split, and this is also true for the R-algebra map that sends $x_i \mapsto x_i^{m_i}$, $1 \le i \le n$. One may send the term $cx_1^{a_1} \cdots x_n^{a_n}$ for $c \in S$ to 0 unless for all i, a_i is divisible by m_i, and to $\alpha(c)x_1^{a_1/m_1} \cdots x_n^{a_n/m_n}$ when $m_i | a_i$ for all i.
(5) In particular, a polynomial ring over a field K is F-split. If α splits $F_K : K \to K$, one may construct a splitting β as follows: for each monomial μ in the x_j, $\beta(c\nu) = 0$ unless $\nu = \mu^p$ is a p th power, and then $\beta(c\mu^p) = F_K(c)\mu$.
(6) The quotient of the polynomial ring $K[x_1, \dots, x_n]$ by an ideal I generated by square-free monomials is F-split. The map β described above induces a splitting.
(7) Similarly, let G be a finite group of permutations of the variables $x_1, \dots, x_n$. The ring of invariants R^G is spanned over K by sums of orbits of monomials. Again, the map β described above induces a splitting. The ring R^G is normal but not necessarily Cohen-Macaulay.
(8) If R is a normal domain of equal characteristic 0, every module-finite extension S of R is split. One can kill a minimal prime of S disjoint from $R - \{0\}$, so that both are domains. Let $\mathcal{K} \hookrightarrow \mathcal{L}$ be the induced map of fraction fields and $\mathrm{tr}_{\mathcal{L}/\mathcal{K}} : \mathcal{L} \to \mathcal{K}$ be field trace. Let $d = [\mathcal{L} : \mathcal{K}]$. Then the restriction of $\frac{1}{d}\,\mathrm{tr}_{\mathcal{L}/\mathcal{K}}$ to S is an R-module retraction $S \to R$, i.e., yields a splitting.
(9) If R is regular of equal characteristic, then every module-finite extension of R is split. See [Ho2]. This is conjectured to be true in mixed characteristic, where it is easy in dimension ≤ 2, known in dimension 3 [Heit], and an open question in dimension ≥ 4.
(10) In particular, in characteristic $p > 0$, every regular ring is F-pure. Let S be a ring of characteristic p. If $I \subseteq S$, and $q = p^e$ is a power of p, then $I^{[q]}$ denotes the ideal $(s^q : s \in I)S$ generated by all q th powers of elements of I (it suffices to use q th powers of generators of I). The following result of Richard Fedder is called *Fedder's criterion* for F-purity: *In characteristic $p > 0$, If (S, m) is regular local, or else a polynomial ring over a field and its homogeneous*

maximal ideal, and I is a proper ideal of S (I is assumed to be homogeneous in the polynomial ring case), then S/I is F-pure if and only if $I^{[p]} : I \not\subseteq m^{[p]}$. Cf. Theorem 1.12 in [Fe].

(11) We can apply Fedder's criterion to understand what happens for the cubical cone $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$ over a field K of characteristic $p > 0$, where $p \neq 3$. Let $S = K[X, Y, Z]$, the polynomial ring. Fedder's criterion asserts that R is F-pure if and only if $(X^3 + Y^3 + Z^3)^p :_S (X^3 + Y^3 + Z^3) \not\subseteq (X^p, Y^p, Z^p)$, i.e., if and only if $(X^3+Y^3+Z^3)^{p-1} \notin (X^p, Y^p, Z^p)$. When we expand the left hand side, a typical term is $\binom{p-1}{i\ j\ k}X^{3i}Y^{3j}Z^{3k}$ where $i+j+k = p-1$. The multinomial coefficient $\binom{p-1}{i\ j\ k} = (p-1)!/i!j!k!$ does not vanish. If p has the form $3h+2$ then at least one of i, j, k is $\geq h+1$, and when we multiply by 3 we get an exponent that is $\geq p$. Hence, Fedder's criterion shows that R is *not* F-pure when $p \equiv 2 \bmod 3$. When $p = 3h + 1$ there is a nonzero term that is multiple of $x^{3h}y^{3h}z^{3h}$, where $i = j = k = h$, and so Fedder's criterion shows that R is F-pure if and only if $p \equiv 1 \bmod 3$.

The following three examples all use the fact that the rings considered are weakly F-regular (and, for that matter, strongly F-regular): see §7 and the results of [Ho1] (for (12)) and [HH5] (for (13) and (14)). Moreover, all of the rings in these three examples split from *every* module-finite extension. See §7 and [HH5].

(12) A normal K-subalgebra R of a polynomial ring $K[X_1, \ldots, X_n]$ such that R is generated over the field K by monomials in the variables $X_1, \ldots, X_n$ is F-split in characteristic p.

(13) If K has prime characteristic $p > 0$, X is an $r \times s$ matrix of indeterminates over K, $1 \leq t \leq \min\{r, s\}$, and $I_t(X)$ denotes the ideal generated by the $t \times t$ minors of X, which is prime (cf. [HE]) then $K[x_{ij}]/I_t(X)$ is F-split.

(14) With the same notation as in (13), If S denotes the subring of $K[x_{ij}]$ generated by the $r \times r$ minors of X, which is the homogeneous coordinate ring of a Grassmann variety, then S is F-split.

When $R \subseteq S$ is pure, it is always true that for every ideal $I \subseteq R$, $IS \cap R = I$: this follows because $R \subseteq S$ remains injective after one applies $R/I \otimes_R \_$. When R is Noetherian, the converse is true under mild conditions on R: see [Ho5].

3. Review of local cohomology

Several of the applications of F-splitting techniques that we discuss in the sequel make use of basic results about local cohomology. In this section we give a brief review of what we need. The reader may consult [GrHa] for a detailed treatment. Let R be a Noetherian ring, I an ideal, and let M be an R-module, which need not be finitely generated. Then we may take as a definition that

$$H^i_I(M) = \varinjlim{}_t \operatorname{Ext}^i_R(R/I^t, M).$$

The ideals I^t may be replaced by any decreasing sequence of ideals cofinal with the powers of I, and these modules depend only on $\mathrm{Rad}(I)$. If $f_1, \ldots, f_h$ generate an ideal with the same radical as I, these modules are also the cohomology of the complex

$$(\dagger)\quad 0 \to M \to \bigoplus_i M_{f_i} \to \bigoplus_{i_1<i_2} M_{f_{i_1}f_{i_2}} \to \cdots$$

$$\to \bigoplus_{i_1<\cdots<i_t} M_{f_{i_1}\cdots f_{i_t}} \to \cdots \to M_{f_1\cdots f_h} \to 0,$$

which is the same as the tensor product of M with the total tensor product of all of the complexes $0 \to R \to R_{f_i} \to 0$. If we omit M and start the numbering with $\oplus M_i$ we have the Čech complex on $U = \mathrm{Spec}(R) - V(I)$ of the sheaf $M^{\sim}|_U$ with respect to the affine open cover given by the sets $D(f_i)$. If $I \subseteq R$, S is a Noetherian R-algebra, and M is an S-module, we may view $M = {}_RM$ as a module over R by restriction of scalars. In this case $H^i_I({}_RM) \cong H^i_{IS}({}_SM)$.

If R and M are $\mathbb{Z}$-graded and I is homogeneous we may choose the $f_1, \ldots, f_n$ to be homogeneous. Every term in the complex $(\dagger)$ is $\mathbb{Z}$-graded, and the maps preserve the grading. Thus, we get a $\mathbb{Z}$-grading on the local cohomology modules that turns out to be independent of which homogeneous generators $f_1, \ldots, f_n$ we choose.

Also note that if M is an R-module, and we denote by ${}^eM$ the R-module obtained by restricting scalars via the map $F^e : R \to R$ (so that for $u \in {}^eM$, the value of $r \cdot u$ is $r^{p^e}u$), then $H^i_I({}^eM) \cong {}^eH^i_I(M)$. To see why, denote by S the target copy of R when one applies F^e. Think of M as an S-module. Then ${}^eM$ is obtained from M by restriction of scalars, and $H^i_I({}_RM) \cong {}_RH^i_{IS}(M) \cong {}_RH^i_(I^{[p^e]}(M) = {}_RH^i_I(M)$, since $I^{[p^e]}$ and I have the same radical.

When R has prime characteristic $p > 0$, there is a natural action of the Frobenius endomorphism F of R on $H^i_I(R)$. One way to think of this is to think of the map $F : R \to R$ as a map $R \to S$, where $S = R$. Then $F : R \to S$ induces a map $H^i_I(R) \to H^i_I(S) \cong H^i_{IS}(S) \cong H^i_{I^{[p]}}(R) \cong H^i_I(R)$ since $I^{[p]}$ has the same radical as I, and this map $F : H^i_I(R) \to H^i_I(R)$ is the action of F that we want. It has the property that $F(ru) = r^pF(u)$ for all $r \in R$ and $u \in H^i_I(R)$. When R is $\mathbb{Z}$-graded and I is homogeneous, the action of F on $H^i_I(R)$ is such that if u is homogeneous of degree $d \in \mathbb{Z}$, then $F(u)$ has degree pd. Hence, $F^e(u)$ has degree p^ed. If $F : R \to R$ splits or is pure, the action of F is injective. This is critically important in the sequel.

Note that every element of every $H^i_I(M)$ is killed by some power of I.

When M is Noetherian and $IM \neq M$, the first nonvanishing $H^i_I(M)$ occurs when $i = d$, the depth of M on I.

Now suppose that M is finitely generated and m is a maximal ideal of R. Then the modules $H^i_m(M)$ are Artinian modules, and since every element is killed by a power of m, they may be viewed as modules over R_m or even over its completion. If (R, m) is local and $M \neq 0$ is finitely generated, then $H^i_m(R)$ is nonzero when i is the depth of M on m and when $i = \dim(M)$. It vanishes if i is smaller than the depth of M or larger than $\dim(M)$. Hence, $M \neq 0$ is Cohen-Macaulay over (R, m) if and only if it has a unique nonvanishing local cohomology module $H^i_m(M)$, which occurs when i is the depth of M on m or, equivalently, the dimension of M.

If (R, m) is regular local (or Gorenstein) of Krull dimension n, then $E = H^n_m(R)$ is an injective hull for the residue class field $K = R/m$. In this case, we have local duality: if M is finitely generated, for all i, $H^i_m(M) \cong \mathrm{Ext}^{n-i}_R(M, R)^\vee$, where $\_^\vee$ denotes $\mathrm{Hom}_R(\_, E)$.

An important consequence of local duality is the following:

Lemma 3.1. *Let (R,m) be a Gorenstein local ring of Krull dimension n and let $M \neq 0$ be a finitely generated R-module of pure dimension d. Suppose that M_P is Cohen-Macaulay for every prime P of R in its support different from m. Then $H^i_m(M)$ has finite length for every $i < d = \dim(M)$. In particular, this holds when $M \neq 0$ is finitely generated and torsion-free over R/Q for some prime Q of R if M is Cohen-Macaulay when localized at any proper prime in its support.*

This follows from the fact that this local cohomology module $H^i_m(M)$ is the Matlis dual of $N = \mathrm{Ext}^{n-i}_R(M, R)$, and so it suffices to show that N has finite length for $i < d$. Since N is finitely generated, we need only show that N is not supported at any prime $P \neq m$ in the support of M. But $N_P \cong \mathrm{Ext}^{n-i}_{R_P}(M_P, R_P)$ which, by Matlis duality over R_P, will vanish if and only if $H^{h-(n-i)}_{PR_P}(M_P) = 0$, where $h = \dim(R_P)$ the height of P. Since M_P is a Cohen-Macaulay module over R_P of pure dimension $h-(n-d)$ (the height of its annihilator does not change when we localize at P, and that height is $n-d$), it has only one nonvanishing local cohomology module, namely $H^{h-(n-d))}_{PR_P}(M_P)$. Since $i < d$, $H^{h-(n-i)}_{PR_P}(M_P) = 0$, as required. □

We also note the following fact, which connects local cohomology with cohomology of sheaves on projective spaces.

Proposition 3.2. *Let K be a field and let R be a finitely generated $\mathbb{N}$-graded K-algebra of Krull dimension n such that $[R]_0 = K$ and R is generated by the vector space $[R]_1$ of forms of degree one. Let M be a finitely generated $\mathbb{Z}$-graded R-module, and let $\mathcal{M}$ denote the corresponding sheaf on $X = \mathrm{Proj}(R)$, so that if $f \in m$, the homogeneous maximal ideal of R, then $\Gamma(X_f, \mathcal{M}) = [M_f]_0$. Then for $i \geq 1$, $H^i(X, \mathcal{M}) \cong [H^{i+1}_m(M)]_0$. More generally, for every $t \in \mathbb{Z}$, $H^i(X, \mathcal{M}(t)) \cong [H^{i+1}_m(M)]_t$.*

If, moreover, R is a domain of positive dimension and M is a nonzero torsion-free R-module then the following conditions are equivalent:

(1) M is Cohen-Macaulay.
(2) $H^i_m(M) = 0$, $0 \leq i < \dim(R)$.
(3) If $n \geq 2$, M has depth at least two on m and for all $t \in \mathbb{Z}$, $H^i(X, \mathcal{M}(t)) = 0$, $1 \leq i < \dim(X)$.

Proof. Let $f_1, \dots, f_n$ be a homogeneous system of parameters for the $\mathbb{N}$-graded ring R, so that $I = (f_1, \dots, f_n)R$ is primary to the homogeneous maximal ideal m. Then $H^\bullet_m(M) = H^\bullet_I(M)$ is the cohomology of the complex (†) displayed in the first paragraph of this section. If we drop the first term of this complex, shift the numbering by one, and take the degree 0 part, we get the Čech complex for computing the cohomology of the sheaf $\mathcal{M}$. This yields that $H^i(X, \mathcal{M}) \cong [H^{i+1}_I(M)]_0$ for $i \geq 1$. The final statement follows if one applies this fact to $M(t)$ (M with the grading shifted so that $[M(t)]_s = [M]_{s+t}$: the sheaf on X corresponding to $M(t)$ is $\mathcal{M}(t)$).

In the graded case, to check that M is Cohen-Macaulay of maximum dimension it suffices to check that $\mathrm{depth}_m M = \dim(R)$, and the depth is the same as the smallest integer d such that $H^d_m(M) \neq 0$. Since $d \leq n$ in any case, we have that (2) is the equivalent to the Cohen-Macaulay property, while (3) is equivalent to (2) by the first part of the proposition. □

4. Proving that rings are Cohen-Macaulay

One of the motivations for studying F-pure and F-split rings is the following fact:

Theorem 4.1. *Let R be a domain that is finitely generated over a field K of characteristic $p > 0$ and that is generated by its forms of degree 1. Suppose that R has depth at least two on m (which holds, for example, if R is normal), is Cohen-Macaulay when localized at a prime other than maximal ideal, and is F-pure. Let $\mathrm{Proj}(R) = (X, \mathcal{O}_X)$. Then R is Cohen-Macaulay if and only if $H^i(X, \mathcal{O}_X) = 0$, $1 \leq i < \dim(X)$.*

Proof. We may assume that $R \neq K$, since K is Cohen-Macaulay, and so $\dim(R) \geq 1$. We know that the depth is at least two, and so it suffices to show that $H^{i+1}_m(R) = 0$ for $1 \leq i < \dim(R) - 2$. Since R is Cohen-Macaulay when localized at any prime P except m, we know that $H^{i+1}_m(R)$ has finite length for all i in the specified range. Hence $[H^{i+1}_m(R)]_t = 0$ whenever $|t| \gg 0$. But the Frobenius endomorphism F and, hence, all of its iterates F^e act injectively on the local cohomology modules since R is F-pure. These modules are $\mathbb{Z}$-graded and $F^e : [H^{i+1}(R)]_t \to [H^{i+1}(R)]_{p^e t}$. The latter vanishes for $e \gg 0$ if $t \neq 0$, and this shows that $[H^{i+1}(R)]_t = 0$ for $1 \leq i \leq \dim(R) - 2$ if $t \neq 0$. But $[H^{i+1}(R)]_0 = H^i(X, \mathcal{O}_X) = 0$ for i in the specified range by hypothesis, and so $[H^{i+1}(R)]_t = 0$ for all t for $1 \leq i \leq \dim(R) - 2$, as required. □

The original proof of the following result, first established in [HR1], utilized a variant of this result. First note that when we say that an algebraic group G *acts rationally* on a K-vector space, we mean that the vector space is a directed union of finite-dimensional G-stable subspaces V such that the group action on V is given by a regular map $G \times V \to V$. For example, if G acts rationally on the vector space of one-forms in a polynomial ring S over K, the action extends uniquely to a rational action of G on S.

Theorem 4.2 (Hochster-Roberts). *Let G be a linearly reductive linear algebraic group over a field K acting rationally, by K-algebra automorphisms, on a Noetherian K-algebra S. Then $R = S^G$, the ring of invariants, is Cohen-Macaulay.*

This is very largely a theorem about equal characteristic 0, because there are very few linearly reductive groups in positive characteristic: there are finite groups of order invertible modulo p, products of $\mathrm{GL}(1, K)$ (called *algebraic tori*) and groups obtained from these by extension. In equal characteristic 0, one has the classical groups (cf. [Weyl]) which have many interesting representations with rings of invariants that are of considerable importance in algebraic geometry. In addition to finite groups and algebraic tori, the semisimple groups (which include the special linear, special orthogonal, and symplectic groups) are linearly reductive.

The proof of the theorem uses the fact that if G is linearly reductive and acts on S as in the theorem, there is a canonical R-module retraction $S \to S^G = R$, called the *Reynolds operator*. But there are some rather subtle points in the argument. Although $R \to S$ is a split extension, this is not true when one passes to characteristic p — it is often false for every p.

For example, let X be a 2×3 matrix of indeterminates and let $A \in \mathrm{SL}(2, \mathbb{Q})$ act on the polynomial ring $\mathbb{Q}[X]$ in these indeterminates by mapping the entries of X to the entries of $A^{-1}X$. Let $\Delta_1, \Delta_2, \Delta_3$ denote the 2×2 minors of X. Then $S^G = \mathbb{Q}[\Delta_1, \Delta_2, \Delta_3]$ is the ring of invariants, and there is an R-module retraction $S \to R$. However, in characteristic $p > 0$, $(\mathbb{Z}/p\mathbb{Z})[\Delta_1, \Delta_2, \Delta_3] \to (\mathbb{Z}/p\mathbb{Z})[X]$ does not split over $(\mathbb{Z}/p\mathbb{Z})[\Delta_1, \Delta_2, \Delta_3]$ for any prime $p > 0$. This means that if one restricts the canonical splitting $\mathbb{Q}[X] \to \mathbb{Q}[\Delta_1, \Delta_2, \Delta_3]$ to $\mathbb{Z}[X]$, it takes on values in such a way that every prime $p \in \mathbb{Z}$ is needed in the denominator in at least one of its values!

In fact, if

$$(\mathbb{Z}/pZ)[\Delta_1, \Delta_2, \Delta_3] \to (\mathbb{Z}/p\mathbb{Z})[X]$$

were split then, if we let $I = (\Delta_1, \Delta_2, \Delta_3)$, the map of local cohomology

$$H^3_I\big((\mathbb{Z}/p\mathbb{Z})[\Delta_1, \Delta_2, \Delta_3]\big) \to H^3_I\big((\mathbb{Z}/p\mathbb{Z})[X]\big)$$

would be injective. Since the former is not 0, this would imply that $H^3_I\big(\mathbb{Z}/p\mathbb{Z})]X]\big) \neq 0$. But this local cohomology module is 0 by a result of Peskine and Szpiro [PS] that we discuss in the next section.

In the original proof of the Hochster-Roberts theorem one uses induction on the dimension to reduce to the case of a supposed counter-example of minimum dimension. One can then pass to associated graded rings to get a counter-example in which G acts linearly on a polynomial ring S over a field. From the minimality, one can assume that R is Cohen-Macaulay except when localized at its homogeneous maximal ideal. One then makes use of reduction to characteristic p. Although one cannot preserve the splitting of $R \to S$ as one passes to characteristic $p > 0$, one can preserve finitely many consequences of the fact that one has a splitting. This is enough to imitate the argument in the characteristic p result stated at the beginning of this section, and thus one is able to show that for $t \neq 0$, the graded components of $[H^i_m(R)]_t$ for $i < \dim(R)$ vanish. One is left with the problem of studying the component in degree 0. Since it is easy to see that R is normal, what one needs to show is that with $\mathrm{Proj}(R) = (X, \mathcal{O}_X)$, one has that $H^i(X, \mathcal{O}_X) = 0$, $1 \leq i < \dim(X)$. Again, one uses reduction to characteristic p, but for this argument one needs the fact that the Frobenius endomorphism is flat in a regular ring of characteristic $p > 0$. In retrospect, the argument given can be seen to be a precursor of tight closure theory, which is discussed in §7.

[Ke] gives a different treatment of the theorem. Boutot [Bou] showed that if R, S are affine algebras over a field of characteristic such that S rational singularities and $R \to S$ is split, then R has rational singularities. There is a brief treatment of rational singularities in [KKMS], pp. 49–52. Boutot's argument uses a characterization of rational singularities in [KKMS] that depends on the Grauert-Riemenschneider vanishing theorem [GR].

Tight closure theory has been used to give substantial generalizations of the Hochster-Roberts theorem: see §7.

Here is another early application of Frobenius splitting ideas to proving that certain rings are Cohen-Macaulay. Let Σ be a finite simplicial complex with vertices $x_1, \ldots, x_n$. This simply means that Σ is a set of subsets of $x_1, \ldots, x_n$ closed under passing to subsets and containing each of the sets $\{x_i\}$. The elements σ of Σ are called *simplices*. The *dimension* of the simplex σ is one less than the number of vertices in σ, and the *dimension* of Σ is the largest dimension of any of its simplices. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. We can establish a bijection of the x_i with the e_i by letting x_i correspond to e_i, $1 \leq i \leq n$, and, hence, between the simplices of Σ and a set of subsets of $\{e_1, \ldots, e_n\}$. The *geometric realization* $|\Sigma|$ of Σ is the topological subspace of $\mathbb{R}^n$ which is the union of the convex hulls of the subsets of $e_1, \ldots, e_n$ corresponding to simplices in Σ. Note that Σ is a compact topological space. The *link* of $\sigma \in \Sigma$ is the simplicial complex consisting of all $\tau \in \Sigma$ disjoint from σ such that $\tau \cup \sigma \in \Sigma$. If $\sigma = \{x_i\}$, the union of all the simplices of Σ that contain x_i is a cone with vertex x_i over the link of $\{x_i\}$.

To a simplicial complex Σ one can associate the *Stanley-Reisner ring* or *face ring* over the field K, $K[x_1, \ldots, x_n]/I_\Sigma$, where I_Σ is generated by all square-free monomials in the indeterminates $x_1, \ldots, x_n$ such that the set of variables that occurs is not a simplex in Σ. The following characterization of when $K[x_1, \ldots, x_n]/I_\Sigma$ is Cohen-Macaulay is given in [Reis]. Note that the *reduced simplicial cohomology* of Σ with coefficients in K agrees with the simplicial cohomology over K in positive degree (the simplicial cohomology is the same as, say, the singular cohomology of $|\Sigma|$ with coefficients in K). In degree 0, if $H^0(\Sigma;\, K)$ has dimension $r > 0$, the reduced simplicial cohomology $\widetilde{H}^0(\Sigma;\, K)$ has dimension $r - 1$, so that it vanishes when $|\Sigma|$ is connected.

Theorem 4.3 (G. Reisner). *Let K be a field, and let Σ be a finite simplicial complex. Then the Stanely-Reisner ring $K[x_1, \ldots, x_n]/I_\Sigma$, where, as above, $K[x_1, \ldots, x_n]$ is a* *polynomial ring in variables corresponding to the vertices of* Σ*, is Cohen-Macaulay if and only if the following two conditions hold:*

(1) *The reduced simplicial cohomology* $\widetilde{H}^i(\Sigma; K)$ *of* Σ *with coefficients in* K *vanishes for* $i < \dim(\Sigma)$*.*
(2) *The reduced simplicial cohomology* $\widetilde{H}^i(\Lambda; K)$ *of every link* Λ *of every simplex of* Σ *vanishes for* $i < \dim(\Lambda)$*.*

This characterization, combined with results of Macaulay on the Hilbert functions of graded Cohen-Macaulay rings, was used by Richard Stanley [St] to prove the Upper Bound Conjecture for simplicial polytopes. Munkres [Mun] showed that Reisner's conditions actually constitute a purely topological condition on Σ.

Sketch of the proof. The case where the field has characteristic 0 can be proved by reduction to characteristic p. The original proof in characteristic $p > 0$ used the fact that Stanley-Reisner rings are F-split. The condition on the links implies, by induction, that the Cohen-Macaulay property holds except possibly at the homogeneous maximal ideal. One can conclude that the local cohomology is of finite length except in the top dimension. There is a $\mathbb{Z}^n$-grading (or grading by monomials) on R/I_Σ, on m, and hence on the local cohomology modules $H^i_m(R/I_\Sigma)$. The action of Frobenius multiplies multi-degrees by p and is injective because of the F-split condition. It follows that any multi-graded component in which any of the n coordinates of the degree is nonzero must vanish. Therefore one can reduce the problem to the vanishing of the local cohomology modules in degree $(0, 0, \ldots, 0)$, and so one can use the degree $(0, 0, \ldots, 0)$ part of the complex displayed in (†) in the first paragraph of §3, with $M = R/I_\Sigma$ and the f_j are taken to be the images of the x_j, to calculate it. This turns out to be the same complex used to calculate the reduced simplicial cohomology of Σ. □

5. Some results of Peskine and Szpiro

The joint work of Peskine and Szpiro in [PS] had an enormous influence: they used techniques involving the application of the Frobenius endomorphism to prove several local homological conjectures due to M. Auslander and H. Bass in characteristic p, introducing conjectures of their own in the process. They also obtained many equal characteristic cases by reduction to characteristic $p > 0$. See also [Ho3], where the existence of big Cohen-Macaulay modules is proved by reduction to characteristic $p > 0$ and then applied to settle the same conjectures in equal characteristic. Many of their results depend on the fact that Frobenius is flat relative to modules of finite projective dimension. (See also [Her].) This means that if we write S for R viewed as an R-algebra via a power F^e (under composition) of the Frobenius endomorphism and M is an R-module of finite projective dimension, then $\mathrm{Tor}_i^R(M, S) = 0$ for all $i \geq 1$. This may be viewed as a generalization of the fact that S is R-flat when R is regular. In fact, the flatness of $F : R \to R$ is equivalent to the regularity of R: cf. [Ku1].

Because of its remarkably simple proof via Frobenius techniques we want to discuss one further result of [PS], which was applied in §4 to show that certain rings of invariants are not direct summands of (nor pure in) polynomial rings in characteristic p.

Theorem 5.1 (C. Peskine and L. Szpiro). *Let R be a regular domain of prime characteristic $p > 0$ and I an ideal of R such that R/I is Cohen-Macaulay. Let h denote the height of I. Then $H^j_I(R) = 0$ for $j > h$.*

Proof. The fact that $F^e : R \to R = S$ is flat implies that $S \otimes_R R/I = R/I^{[p^e]}$ is Cohen-Macaulay for all e. But then there is a unique nonvanishing $\mathrm{Ext}^j_R(R/I^{[p^e]}, R)$ for all e, occurring when $j = h$. Since the local cohomology may be obtained as the direct limit of these, it follows that $H^j_I(R) = 0$ except when $j = h$. □

6. Small Cohen-Macaulay modules

It is known (cf. [Ho3, HH4, HH6]) that over every equal characteristic local ring (R, m), there is a module (even an algebra) B such that $mB \neq B$ and every system of parameters for R is a regular sequence on B. B is called a *big Cohen-Macaulay* module (respectively, algebra) for R. This was first proved by reduction to characteristic p in [Ho3], and all known proofs require reduction to characteristic p. This is an open question in mixed characteristic in dimension 4 and higher. (The dimension 3 case is settled using the results of [Heit] in [Ho6].)

It has long been an open question whether, under mild conditions on a local ring (R, m) (e.g., if R is excellent), there exists a Cohen-Macaulay module that is finitely generated (hence, the use of the word "small") whose dimension is the same as $\dim(R)$. In this section we give an application of Frobenius splitting techniques to proving the existence of small Cohen-Macaulay modules in characteristic $p > 0$ in certain instances. The argument was first given by R. Hartshorne and later rediscovered independently first by C. Peskine and L. Szpiro and later by the author. (The argument is given, for example, in [Ho4].) For simplicity, we have not attempted to state the most general form of the result here. But the question remains open even for $\mathbb{N}$-graded affine algebras over a field of characteristic 0 in dimension 3, and it is an open question for local rings of affine algebras over a field of characteristic $p > 0$ in dimension 3.

Theorem 6.1 (Hartshorne). *Let R be a finitely generated $\mathbb{N}$-graded domain over a perfect field K of characteristic $p > 0$ with $[R]_0 = K$. Let M be a finitely generated $\mathbb{N}$-graded R-module that is torsion-free over R, and suppose that M_P is Cohen-Macaulay over R_P for every prime ideal P of R except possibly the homogeneous maximal ideal m. Then R has a graded finitely generated module N that has depth equal to the dimension of R.*

Sketch of the proof. We may assume that R has positive dimension and is graded so that $[R]_i \neq 0$ for all $i \gg 0$, and then the same will be true for M. The fact that M_P is Cohen-Macaulay for $P \neq m$ implies, that the local cohomology modules $H^i_m(M)$ have finite length for $i < d = \dim(R)$ by the Lemma in §3 (R is a homomorphic image of a Gorenstein ring). Let $F^e : R \to R$, and consider M as module over the right hand copy of R. Restriction of scalars produces a module ${}^eM$ over the left hand copy of R as in the fourth paragraph of §3. The grading on M enables us to split ${}^eM$ into the direct sum of p^e nonzero R-modules N_j, $0 \leq j < p^e$, where

$$N_j = \bigoplus_{i \equiv j \bmod p^e} [M]_i.$$

Let B denote the sum of the lengths of the $H^i_m(M)$ for $i < \dim(R)$. We claim that for all e so large that $p^e > B$, at least one of the modules N_j is Cohen-Macaulay. For consider the sum of the lengths L_j of the local cohomology modules $H^i_m(N_j)$ for $i < \dim(R)$. All we need to show is that at least one L_j is 0. But the total of the L_j is the same as the sum of the lengths of the $H^i_m({}^eM)$ for $i < \dim(R)$, and, as noted in §3, $H^i_m({}^eM) = {}^eH^i_m(M)$, and, because K is perfect, this has the same length as $H^i_m(M)$. But then $\sum_{j=0}^{p^e-1} L_j = B$. Since $p^e > B$, at least one of the L_j must be zero. □

7. Tight closure and splinters

We give here the very briefest introduction to tight closure theory, which has many interconnections with questions about F-splitting and F-purity.

Throughout this section, R is an excellent ring. In characteristic $p > 0$, u is defined to be in the *tight closure* of an ideal I of R if there is an element $c \in R$ not in any minimal prime such that $cu^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. This holds if and only if it holds modulo every minimal prime of R. We focus primarily on the case where R is a domain. In that case, c is simply required to be nonzero. For the characteristic $p > 0$ theory see [HH1, HH2, HH3, HH5, HH7] and [Sm].

Tight closure may also be defined in finitely generated $\mathbb{Q}$-algebras as follows: if R is such an algebra, $u \in R$, and $I \subseteq R$ we say that u is in the *tight closure* of J in R if there is a domain $R_0 \subseteq R$ finitely generated over $\mathbb{Z}$ such that $u \in R_0$, and an ideal $I \subseteq J \cap R_0$ such that the image of u is in the tight closure of IR_0/pR_0 in R_0/pR_0 for all but finitely many prime integers p. One can then extend the theory to all excellent Noetherian rings containing $\mathbb{Q}$ as follows: u is in the tight closure of J in R if there exists a finitely generated $\mathbb{Q}$-algebra A and ideal $I \subseteq A$, an element $t \in A$ in the tight closure of I, and a homomorphism $A \to R$ such that $t \mapsto u$ and I maps into J. This notion is called *equational tight closure* in [HH7].

There is also a tight closure theory for submodules of modules.

A ring such that every ideal is tightly closed is called *weakly F-regular*. If all localizations of R are weakly F-regular, R is called *F-regular*. It is not known whether weakly F-regular implies F-regular for excellent rings.

In the equicharacteristic case, one has the following for excellent rings:

(1) Every ideal of a regular ring is tightly closed.
(2) If $x_1, \ldots, x_k$ is part of a system of parameters in a reduced equidimensional local ring and $rx_k \in (x_1, \ldots, x_{k-1})$, then r is in the tight closure of $(x_1, \ldots, x_{k-1})$.
(3) If R is weakly F-regular, then R is Cohen-Macaulay.
(4) If $R \to S$ is pure and S is weakly F-regular, then so is R.
(5) If $R \subseteq S$ is an integral extension, $IS \cap R$ is contained in the tight closure of I.
(6) If R is weakly F-regular, then R is normal.

These results imply that in the equicharacteristic case, every ring R pure in a regular ring is Cohen-Macaulay. This is a generalization of the Hochster-Roberts theorem discussed in §4. This is an open question in the mixed characteristic case.

We refer to a Noetherian ring that is a direct summand of every module-finite extension ring as a *splinter*. The results of [HH5] (see Corollary 5.23 and Theorem 5.25 on p. 630) coupled with the results of [Ho5] imply that every weakly F-regular ring is a splinter, and, hence, F-pure. In the Gorenstein case, in characteristic $p > 0$, the converse is true: splinters are weakly F-regular. This is also true in the $\mathbb{Q}$-Gorenstein case (cf. [Si]). In general, it is known that in positive characteristic a splinter must be Cohen-Macaulay, but it is an open question whether splinters are weakly F-regular in general in the Cohen-Macaulay case.

A different point of view connecting splitting questions with tight closure in the characteristic $p > 0$ case is the following. Let S be a module-finite extension of a reduced ring R of characteristic $p > 0$. From the point of view of Yoneda Ext, the exact sequence

$$0 \to R \to S \to S/R \to 0$$

of finitely generated R-modules represents an element ϵ of $E = \mathrm{Ext}^1_R(S/R,\, R)$. If we compute E using a finite projective resolution $P_\bullet$ of S/R, then E may be viewed as a submodule of $Q = \mathrm{Hom}_R(P_1,\, R)/\mathrm{Im}\big(\mathrm{Hom}_R(P_0,\, R)\big)$. Theorem 5.17 of [HH5] yields:

Theorem 7.1. *With notation and hypotheses as in the paragraph just above, the element* $\epsilon \in \mathrm{Ext}^1_R(S/R,\, R)$ *represented by* $0 \to R \to S \to S/R \to 0$ *is in the tight closure of 0 when regarded as an element of* Q*. Hence, if* R *is weakly F-regular,* ϵ *is 0, and* $R \hookrightarrow S$ *splits.*

This yields a proof from a different perspective of the fact that weakly F-regular rings are splinters.

Tight closure is connected with Frobenius splitting in another way. Let R be Noetherian of characteristic $p > 0$. R is called *F-finite* if $F : R \to R$ is module-finite. F-finite rings are excellent (cf. [Ku2]). An F-finite domain of characteristic p is called *strongly F-regular* if for every $c \neq 0$, the map $R \to R^{1/p^e}$ such that $1 \mapsto c^{1/p^e}$ splits for all sufficiently large e. See [HH1] and [HH3]. It is easy to show that strongly F-regular rings are F-regular. In the F-finite Gorenstein case, weakly F-regular is equivalent to strongly F-regular. The converse is an open question in the general case.

The rings discussed in Examples (12), (13), and (14) of §2 are known to be strongly F-regular (see [Ho1] for Example (12), and [HH5], Theorem 7.14, p. 651 for Examples (13) and (14): note that strong F-regularity follows from weak F-regularity in these cases because the rings are either Gorenstein, or algebra retracts of F-regular Gorenstein rings), and so split from every module-finite extension. One can then deduce immediately that all of these rings are F-split.

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