Finiteness of Associated Primes of Local Cohomology Modules

by

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CHAPTER I

Introduction

The main objects of study in this thesis are local cohomology modules. We write $H^i_I(M)$ for the ith local cohomology of a module M with respect to some ideal I. In this thesis we will look only at the local cohomology of finitely generated modules unless otherwise stated. This cohomology theory gives invariants to help measure many important properties in commutative algebra as well as algebraic geometry.

If we consider the spectrum of a ring as a scheme, or restrict attention to an affine subscheme of something larger, it is often easier to define sections or functions on an open subset then on the whole space. The local cohomology modules can be viewed as sheaf cohomology on the complement of the closed set cut out by the ideal involved. This means elements of the first local cohomology module represent obstructions to extending sections across the whole space. In particular, having the first local cohomology vanish means we can define sections on the open set away from the zero set of our ideal and they always extend to the whole scheme.

One algebraic invariant measured by the local cohomology modules is the depth of a ring or module on an ideal. The depth of M on I can be defined as the smallest positive i for which $H^i_I(M)$ is nonzero. We can also measure depth with other modules such as the Ext modules, but the local cohomology only depends on the radical of I, which gives more flexibility.

The two main problems with studying local cohomology are that it is very hard to compute explicitly and that it is usually very large. There are various approaches to computing these modules, but to give explicit elements is usually difficult. Even giving a set of generators is difficult, because most local cohomology modules are not finitely generated. However, even though they aren't finitely generated modules, local cohomology modules often exhibit finite-like properties.

One such property, finiteness of the sets of associated primes of local cohomology, is the main focus of this thesis. For any R-module, we define the associated primes as follows: $P \in \operatorname{Spec}(R)$ is called an *associated prime* of an R-module M if P is the annihilator of some element $u \in M$. The set of all such primes is called the *assassinator* in R of M, written $\operatorname{Ass}_R M$.

For finitely generated R-modules, this set is always finite, but if M is not finitely generated it is usually infinite. However even when they are not finitely generated, it is possible for the local cohomology modules to have only finitely many associated primes.

One of the major cases where the local cohomology modules are known to have finite assasinators is for equal characteristic or unramified mixed characteristic regular local rings. This has been extended to regular domains finitely generated as algebras over algebraically closed fields of characteristic zero, see [Lyu93, Remark 2.9], and to the completion of a polynomial ring over a mixed characteristic DVR at a maximal ideal containing the maximal ideal of the DVR, see [Lyu00b, Theorem 2]. If our ideal, m, is maximal in $\operatorname{Spec}(R)$, it is also clear that $\operatorname{Ass}_R H^i_m(M)$ is finite since it is supported only at m.

In Chapter 2, we briefly review the background material needed to understand

and prove the results of the later chapters.

Rings of Small Dimension

One way to control the behavior of local cohomology modules is to restrict attention to the case where the ring has small dimension. There, it is possible to prove theorems not only about the local cohomology of the ring itself, but about the local cohomology of its finitely generated modules as well. In [Mar01], Marley shows that local rings of dimension three have only finitely many primes associated to the local cohomology of any finitely generated module with respect to any ideal. He also gets some results for local rings of dimensions 4 and 5 under various additional conditions.

Unfortunately, such results do not hold in rings of larger dimension as shown by examples in a paper of Singh and Swanson. In Theorem 4.1 and Remark 4.2 of [SS04], they show that the ring

$$R = \frac{k[s,t,u,v,x,y]}{(su^2x^2 + tuxvy + sv^2y^2)},$$

has infinitely many primes associated to $H^2_{(x,y)}(R)$. This remains true after localizing at the homogeneous maximal ideal (s,t,u,v,x,y) or specializing to the case where $s = 1$, i.e. modulo $s - 1$. (Note that after setting s equal to 1 we are no longer in a homogeneous situation so we cannot localize and preserve the associated primes.) Since R has dimension 5, this gives examples of a five dimensional local ring and a four dimensional non-local ring whose local cohomology has infinitely many associated primes.

The open cases between Marley's theorems and the Singh and Swanson examples are thus local rings of dimension 4 and non-local rings of dimension 3.

The results of Chapter 3 were inspired by attempts to generalize some of Marley's results to these open cases. Specifically, in Theorem III.5 we prove that if a three dimensional ring has an S_2-ification for every prime cyclic module then $\mathrm{Ass} H^i_I(M)$ is finite for every finitely generated module M and ideal I of height at least two modulo every minimal prime of M.

Adjoining Indeterminates

For rings of small dimension we know that the local cohomology of any finitely generated module has only finitely many associated primes, so, in particular, we know $\mathrm{Ass}_R H^i_I(R)$ is always finite. If R is a regular local ring of equal characteristic or unramified mixed characteristic, it is known that all local cohomology modules of the ring itself have only finitely many associated primes. Since we can control the local cohomology in these good cases, one obvious question is how to extend this good behavior to related rings. Adjoining variables, either as polynomials or power series, is such a well behaved process that it is reasonable to hope that if local cohomology behaves well over the base ring it will also behave well over the new ring formed by adjoining indeterminates. The case where the base ring has small dimension is already quite interesting, because, while the base ring's local cohomology is controlled because the dimension of the ring is small, by adjoining variables we can make the dimension of the polynomial ring and its singular locus both arbitrarily large.

In Chapter 4, we give some cases where adjoining indeterminates preserves the finiteness of the assasinator of our local cohomology modules.

Our first class of base rings consists of those that have resolutions of singularities which can be covered a small number of open affines. This allows us to link the good behavior of the regular rings covering the smooth space resolving $\mathrm{Spec}(R)$ to the behavior of our base ring.

More precisely, in Theorems IV.2 and IV.4 we take a base ring A which has a blowup, Y_0, of A along an ideal of depth at least 2 where Y_0 is covered by 2 (resp. 3) affine patches and all cohomology of the structure sheaf $\mathcal{O}_{Y_0}$ has finite length over A. We show that polynomial and power series rings, R, over such a base have finitely many primes associated to each $H^i_I(R)$.

We also show that if one adjoins finitely many variables to an unramified regular local ring of mixed characteristic, one still has that $\mathrm{Ass}_R H^i_I(R)$ finite even though the new ring is no longer local.

Calm Extensions

After thinking about polynomial and power series extensions, we were inspired to ask what special properties of these extensions allow us to relate the behavior of the local cohomology of our extension ring to that of the base ring. One interesting property polynomial and power series extensions share is that both are flat. For flat extensions $R \to S$, we know that the associated primes of $M \otimes_R S$ are directly controlled by the associated primes of M over R. In Chapter 5 we take a similar statement as the definition of a new class of extensions which we call *calm*. Specifically, $R \to S$ is calm if for each prime P of R, we can find a set $\mathfrak{a}(P) \subset \mathrm{Spec}(S)$ so that for any R-module M

$$\mathrm{Ass}_S S \otimes_R M \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P).$$

If all extensions of a ring are calm we call the ring *serene*.

Chapter 5 explores some properties of these extensions, and gives some basic classes of serene rings. It concludes by exhibiting examples which give some bounds on how far the calm property extends.

CHAPTER II

Background

Before we jump into the body of this thesis, we will give some of the basic theory of local cohomology which underlies our work, as well as some of the tools used to control and understand it.

Unless otherwise specified, all rings are assumed to be commutative with identity and Noetherian, and all modules are assumed to be unital.

2.1 The Local Cohomology Modules

If $I \subset R$ is an ideal and M is an R-module, we let $\Gamma_I(M)$ be the set $\{u \in M | uI^a = 0 \text{ for some positive integer } a\}$. These may be thought of as global sections of the sheaf $\widetilde{M}$ on $\mathrm{Spec}(R)$ corresponding to sections supported on $V(I)$. We define the ith local cohomology module of I on M, $H^i_I(M)$, to be the ith right derived functor of Γ_I applied to M. To compute this directly, apply Γ_I to any injective resolution of M and then take cohomology.

There are a number of alternate formulations of $H^i_I(M)$, including as the directed limit of either Ext modules, $\varinjlim_a \mathrm{Ext}^i_R(R/I^a, M)$, or Koszul cohomology, $\varinjlim_a H^i(f_1^a, \ldots, f_n^a; M)$ where $I = (f_1, \ldots, f_n)$. To compute $H^i_I(M)$ one can also use the Čech complex. If $I = (f_1, \ldots, f_n)$, then $H^i_I(M)$ is the cohomology at the ith spot of the complex

$$0 \to M \to \oplus M_{f_i} \to \oplus M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0.$$

Below are some basic facts about these local cohomology modules. For proofs we refer the reader to Brodmann and Sharp's book [BS98] as well as Hartshorne's lecture notes on Grothendieck [Gro66].

One very nice property of local cohomology is that it has some flexibility in the choice of ideal.

Proposition II.1. *If $\sqrt{J} = \sqrt{I}$ then $H^i_J(M) = H^i_I(M)$ for any i and M.*

Proposition II.2. *For any ideal $I \subset R$ and R-module M, we have $H^i_I(M) = 0$ if any of the following hold:*

(1) $i > dim(R)$

(2) $i > dim(M)$

(3) $i < depth_I(M)$

(4) $i >$ least number of generators of some ideal J with $\sqrt{J} = \sqrt{I}$

Proposition II.3. *Let R be Noetherian, M a finitely generated R-module, and $I \subset R$ an ideal. If $d = depth_I M$, then $Ass_R H^d_I(M) = Ass_R Ext^d_R(R/I, M)$. Since $Ext^d_R(R/I, M)$ is finitely generated, this shows $Ass_R H^d_I(M)$ is finite.*

This means the first non-vanishing local cohomology module has a finite assassinator. The next results of Marley give some information about the last non-vanishing local cohomology modules.

Proposition II.4. *[Mar01, Prop. 2.3] Let $Supp^i_R(M) := \{P \in Supp_R(M) | ht(P) = i\}$. For any finitely generated R-module M and ideal $I \subset R$, $Supp^i_R(H^i_I(M))$ is finite. Since $Ass_R H^i_I(M) \subseteq Supp_R(H^i_I(M))$, this means the ith local cohomology module has only finitely many associated primes of height i.*

Since $H^i_I(M)_P \cong H^i_{PR_P}(M_P)$, two immediate corollaries of this are:

Corollary II.5. *[Mar01, Cor. 2.4] If $dim(R) = n$, then $Supp_R(H^n_I(M))$, and hence $Ass_R H^n_I(M)$, is finite for any ideal I.*

Corollary II.6. *[Mar01, Cor. 2.5] If $dim(R) = n$ and R is local, then $Supp_R(H^{n-1}_I(M))$, and hence $Ass_R H^{n-1}_I(M)$, is finite for any ideal I.*

Like other cohomology theories, local cohomology also has a number of long exact sequences.

For any two ideals $I, J \subset R$ and R-module M, we get a long exact sequence

$$0 \to H^0_{I+J}(M) \to H^0_I(M) \oplus H^0_J(M) \to H^0_{I\cap J}(M) \to H^1_{I+J}(M) \to \cdots$$

Any short exact sequence, $0 \to N \to M \to M' \to 0$, induces a long exact sequence

$$0 \to H^0_I(N) \to H^0_I(M) \to H^0_I(M') \to H^1_I(N) \to \cdots$$

Also for any element $f \in R$, we get another long exact sequence

$$0 \to H^0_{I+(f)}(M) \to H^0_I(M) \to H^0_I(M)_f \to H^1_{I+(f)}(M) \to \cdots$$

For any ring, since $H^0_I(M) \subseteq M$, it is clear that the zeroth local cohomology of a finitely generated module is always finitely generated and hence has finite assasinator. From the previous proposition we can show that the first local cohomology module always has finitely many associated primes as well.

Proposition II.7. *$H^1_I(M)$ has finitely many associated primes for any ideal I and finitely generated module M.*

Proof. Let $N = M/H^0_I(M)$. Because $H^i(H^0_I(M)) = 0$ for $i \geq 1$, the long exact sequence arising from $0 \to H^0_I(M) \to M \to N \to 0$ forces $H^i_I(M) \cong H^i_I(N)$ for $i \geq 1$. But no element of I is a zero-divisor on N, so $\text{depth}_I N \geq 1$ and Proposition II.3 finishes the proof. $\square$

2.2 Double Complexes and Spectral Sequences

As in other situations dealing with homology or cohomology of complexes, it is often helpful to form a double complex and use it to link two different computations, as is done when proving $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$.

Given a double complex $A^{\bullet\bullet}$, we can filter its total complex by forming for each p the double subcomplex $\langle A^{\bullet\bullet}\rangle_p$ which is our original double complex $A^{\bullet\bullet}$ where the ith row is replaced by zeros if $i \leq p$. The total complex of this subcomplex, $\mathcal{T}^\bullet(\langle A^{\bullet\bullet}\rangle_p)$, is a subcomplex of the original total complex, $\mathcal{T}^\bullet(A^{\bullet\bullet})$. The set of these subcomplexes gives a filtration of the total complex which gives rise to a spectral sequence as follows.

Let $E_0^\bullet$ be the associated graded complex of $\mathcal{T}^\bullet(A^{\bullet\bullet})$ with respect to the filtration given above. This means E_0 is just the direct sum of the rows of the original double complex. There is an obvious differential here induced by the row maps from $A^{\bullet\bullet}$ which we denote $d^0 : E_0^{p,q} \to E_0^{p,q+1}$. We take $E_1^\bullet$ to be the cohomology of $E_0^\bullet$ with respect to this differential.

We have differentials on this complex induced by the column maps of $A^{\bullet\bullet}$, and take $E_2^\bullet$ to be the cohomology of $E_1^\bullet$.

In general, $E_{r+1}^\bullet$ is the cohomology of $E_r^\bullet$ taken with respect to the differential $d^r : E_r^{p,q} \to E_r^{p+r,p-r+1}$ where $E_r^n = \bigoplus_{p+q=n} E_r^{p,q}$. One can think of this map as going up r rows and left $r-1$ columns in the double complex $E_r^{p,q}$. These do give maps from E_r^n to E_r^{n+1} because they all map from one diagonal to the next since $(p+r)+(q-r+1) = p+q+1$.

These $E_r^\bullet$'s converge to an associated graded complex of $\mathcal{T}^\bullet(A^{\bullet\bullet})$, usually denoted $E_\infty^\bullet$, and we can often compare its properties with those of $\mathcal{T}^\bullet(A^{\bullet\bullet})$.

It is also possible to start filtering the double complex $A^{\bullet\bullet}$ by setting some columns equal to zero. In an analogous way, this gives another spectral sequence converging to another associated graded complext of $\mathcal{T}^{\bullet}(A^{\bullet\bullet})$, albeit with respect to a different grading.

In this way it is often possible to compare the modules of our two filtrations by comparison of their respective $E_{\infty}^{\bullet}$ complexes.

2.3 D-Modules

Since one of the main problems with local cohomology modules is that they are so large, one strategy is to introduce additional structures over which the local cohomology is "smaller". In rings of characteristic 0, Lyubeznik has applied the theory of D-modules in [Lyu93]. These D-modules are basically rings of differential operators over R.

For this section we will be working only with rings of equal characteristic 0, hence containing a field. (The case where the ring is of mixed characteristic is much harder although we will prove a result there in Section 4.2.)

Let R be a k-algebra, where $k \subset R$ is a field of characteristic 0. Then $D = D(R, k)$ is the ring of k-linear differential operators mapping R to R. Our main case is when $R = k[[x_1 \ldots x_n]]$ or $k[x_1, \ldots x_n]$ is a power series or polynomial ring over k. For these rings, D is a free left or right R module generated by all monomials in the $\partial_1 = \frac{\partial}{\partial x_1}, \ldots, \partial_n = \frac{\partial}{\partial x_n}$. The ring D is generated over R by the ∂_i's and all relations are spanned by those of the form $\partial_i\partial_j = \partial_j\partial_i$, $\partial_i x_j = x_j\partial_i$ if $i \neq j$, and $\partial_i x_i - x_i\partial_i = 1$. In fact, the associated graded ring of D is a polynomial ring in n variables over R.

These rings of differential operators are no longer commutative, but may retain some good properties. Via the homomorphism $R \to D$ defined by sending r to the map on R which is multiplication by r, D has an R-algebra structure. Also, since elements of D naturally act on R, we see that R has a D-module structure.

If we have an action of D on an R-module, M, we can extend that action to the localization, M_S, at any multiplicative system S in R via the quotient rule, and localization maps respect the D-module structure.

Since R is a D-module and the Čech complex of R with respect to an ideal I is composed solely of localizations and localization maps, the Čech complex is made up of D-modules connected by D-module maps. Viewing the modules $H^i_I(R)$ as cohomology of this complex gives makes them D-modules as well.

When using this structure to study local cohomology, there is a distinguished class of D-modules to consider. To define them we introduce some notions of dimension for D-modules.

In general if we have a ring A (not necessarily commutative or Noetherian) with a filtration which makes the associated graded ring, $\mathrm{gr}(A)$, commutative and Noetherian, then every finitely generated A-module, M, has some filtration for which $\mathrm{gr}(M)$ a finite $\mathrm{gr}(A)$-module. Let $d(M) = \dim(\mathrm{gr}(M))$.

If our associated graded ring is also regular, we can also define the *weak global dimension*, $wgd(A)$, as the smallest integer, a, where we have $\mathrm{Tor}_i^A(M, N) = 0$ for all M, N if $i > a$. (If A is itself regular, commutative and Noetherian, this is just the usual dimension.)

Going back to our D-module setup, we see that the associated graded ring of D is nice enough to permit the definitions above (at least in the cases we will be interested in). Now we can define the following special class of D-modules.

Definition 1. *A D-module, M, is called* holonomic *if it is finitely generated over D and $d(M) = wgd(gr(A)) - wgd(A)$.*

In the case of a power series ring in n variables over a field, this means the holonomic D-modules are those finitely generated modules with $d(M) = n$.

The next proposition lists some of the basic properties of holonomic D-modules. For a more in depth discussion see [Lyu93] Remarks 2.2a through 2.2f and Remark 2.9.

Proposition II.8. *Let R be a regular domain finitely generated over an algebraically closed field of characteristic 0. Then we have the following:*

(a) R is a holonomic D-module.

(b) If M is holonomic and $f \in R$, then M_f is holonomic.

(c) D-submodules, quotients and extensions by holonomic D-modules are again holonomic.

(d) The local cohomology of a holonomic D-module with respect to any ideal is holonomic.

(e) Holonomic modules have finite filtrations by simple (holonomic) D-modules.

(f) Simple holonomic D-modules have a single associated prime over R.

Because they are finitely generated over D, holonomic modules always have finite assasinator over D, but properties (e) and (f) combine to tell us that holonomic D-modules also have finite sets of associated primes over R. By property (d), this means that to show $\mathrm{Ass}_R H^i_I(M)$ is finite it is enough to show that the module M is holonomic as a D-module.

Lyubeznik uses this fact to show the following, see [Lyu93] Theorem 3.4 (c).

Proposition II.9. *If R is a regular k-algebra where $char(k) = 0$, then for each maximal ideal, m, the set $\{P \subseteq m | P \in Ass(H^i_I(R))\}$ is finite for each i, I.*

This of course means that $\mathrm{Ass}_R H^i_I(R)$ must be finite whenever R is local or appropriately graded.

2.4 Algebraic Geometry

Although the results in this thesis make no reference to algebraic geometry, some of the proofs in Chapter 4 rely heavily on geometric techniques and constructions. Here we outline the material used there, although for a more thorough introduction see Shafarevich [Sha94] or Hartshorne [Har77].

For any ring, R, we can view $X = \mathrm{Spec}(R)$ as an affine scheme via the Zariski topology. Any ideal, $I = (f_1, \ldots, f_n) \subset R$, then corresponds to a closed subscheme, $V(I)$, of X. The complementary open set $U = X - V(I)$ is covered by the open sets $D(f_i)$ each of which is an open affine subscheme when viewed as $\mathrm{Spec}(R_{f_i})$. Select any R-module, M, and form the corresponding quasicoherent sheaf $\widetilde{M}$. On each $D(f_i)$ this sheaf restricts to M_{f_i} with restrictions to overlaps of the open cover given by further localizations.

In this situation we can compute the sheaf cohomology of $\widetilde{M}$ on U by taking cohomology of the scheme-theoretic Čech complex with respect to our open cover of U by the $D(f_i)$'s from

$$0 \to \oplus M_{f_i} \to \oplus M_{f_i f_j} \to \cdots \to M_{f_1 \cdots f_n} \to 0.$$

This is just our usual Čech complex with the zeroth term omitted, which means we have the exact sequence

$$0 \to H^0_I(M) \to M \to H^0(U; \widetilde{M}) \to H^1_I(M) \to 0$$

and

$$H^i_I(M) \cong H^{i-1}(U; \widetilde{M}) \text{ for } i \geq 2.$$

This link between sheaf cohomology and local cohomology gives us a way to apply tools from algebraic geometry to the study of local cohomology modules.

The main technique from algebraic geometry that we use in this thesis is resolution of singularities. Given a scheme, X, a resolution of singularities, Y, for X is a proper birational morphism $\vartheta : Y \to X$ where Y is nonsingular. We are interested in this because Y can be covered by open affines which are the spectra of regular rings giving us much better behavior of the sheaf (and hence the local) cohomology. Because of the good properties of the map, we are sometimes able to relate the cohomology of the smooth space to that of X.

One way to form a resolution of singularities, which we will use in Chapter 4, is by blowing-up. Here one expands a closed set in X by replacing it with its space of tangent directions. This can be used to remove nonsmooth points of X, particularly in the case where $X = \mathrm{Spec}(R)$ and the closed set is defined by the prime ideals P where R_P is not regular.

CHAPTER III

Rings of Small Dimension

In this chapter we show that if a ring has small enough dimension, the local cohomology of any finitely generated module with respect to many ideals has only finitely many associated primes. These results were largely inspired by, and some generalize parts of, Marley's paper [Mar01].

The first section gives some results on the finiteness of associated primes when local cohomology is taken with respect to an ideal of height at least 2. These theorems cover the local cohomology of any finitely generated module. Since the counter-examples of Singh and Swanson, [SS04], use ideals of height 1, there is hope that this could hold for rings beyond dimension 4.

In the second section we look at the case where the ideal has height at most 1, and show that the case of a local four-dimensional ring reduces to an extremely concrete and special situation.

3.1 Ideals of height at least two

In this section, we consider local cohomology modules, $H^i_I(M)$, where $\mathrm{ht}(I) \geq 2$ modulo every associated prime of M. The main tool in these proofs is reducing to a situation where I has depth at least two instead of just height at least two. Our main tool is to find a bigger ring containing our ring R where ideals of height at least two automatically have depth at least two as well. Such a ring is called an S_2-ification for R.

If R is an excellent domain, let $R^\star$ be the ring of all elements from $\mathrm{Frac}(R)$ which are multiplied into R by an ideal of height at least two. This ring $R^\star$ is an S_2-ification for R.

Proposition III.1. *Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finitely generated R-module. If there is a map of R-modules, $\theta : M \to N$, where $depth_I N \geq 2$ and $dim(ker(\theta)) \leq 1$ then $Ass_R H_I^2(M)$ is finite.*

Proof. Let $K = \ker(\theta)$, $M' = M/K$ and $C = N/M'$. We have a short exact sequence $0 \to M' \to N \to C \to 0$, which induces the long exact sequence

$$\cdots \to H_I^1(N) \to H_I^1(C) \to H_I^2(M') \to H_I^2(N) \to \cdots$$

Since $\mathrm{depth}_I N \geq 2$ we know $H_I^1(N) = 0$, so this sequence becomes

$$0 \to H_I^1(C) \to H_I^2(M') \to H_I^2(N) \to \cdots$$

By Proposition II.7 we know $\mathrm{Ass}_R H_I^1(C)$ is finite, and $\mathrm{Ass}_R H_I^2(N)$ is finite by Proposition II.3, so $\mathrm{Ass}_R H_I^2(M') \subseteq \mathrm{Ass}_R H_I^1(C) \cup \mathrm{Ass}_R H_I^2(N)$ forces $\mathrm{Ass}_R H_I^2(M')$ to be finite as well.

Consider the short exact sequence $0 \to K \to M \to M' \to 0$ which induces

$$\cdots \to H_I^1(M') \to H_I^2(K) \to H_I^2(M) \to H_I^2(M') \to \cdots$$

Since $\dim(K) < 2$, we know $H_I^2(K) = 0$. Thus $H_I^2(M) \subseteq H_I^2(M')$ so $\mathrm{Ass}_R H_I^2(M)$ is finite.

□

Before we get to our first main result we need the following two lemmas.

Lemma III.2. *Let M be any finitely generated R-module and $x, y \in R$ be non-zerodivisors on M. Then x, y forms a possibly improper regular sequence on $M' = \{u \in M_{xy} | x^N u, y^N u \in M$ for some $N\}$.*

Proof. Since x is a non-zerodivisor on M, it is clear that x is also a non-zerodivisor on M_{xy} and hence on M'. Suppose that we have $ux = vy$ for some $u, v \in M'$. Let $f = \frac{v}{x} = \frac{u}{y}$ in $(M')_{xy}$. Then $xf, yf \in M'$, so can find some N for which $x^N xf, y^N yf = x^{N+1} f, y^{N+1} f \in M$. Thus $f \in M'$, so $v = f \cdot x \in xM'$ which means x, y is a regular sequence on M'. □

Lemma III.3. *Let R be a ring which has an S_2-ification for R/P whenever $P \in Ass_R M$, and pick $x, y \in R$ so that $ht(x, y)R/P = 2$ for all $P \in Ass_R M$. Then M', defined as in Lemma III.2, is finitely generated as an R-module.*

Proof. We first claim that $M'/M \cong H^1_{(x,y)}(M)$, meaning it will be enough to show that $H^1_{(x,y)}(M)$ is finitely generated.

To see this, consider the Čech complex

$$0 \to M \to M_x \oplus M_y \to M_{xy} \to 0.$$

For any element $u \in M'$, we can write $u = m_1/x^N = m_2/y^N$ for some $m_1, m_2 \in M$ and some N. We can therefore identify elements of M' with elements of $M_x \oplus M_y$ by sending u to $(m_1/x^N, m_2/y^N)$. Such elements are clearly in the kernel of the map to M_{xy}. Any element $\nu = (m_1/x^a, m_2/y^b)$ in the kernel has $m_1/x^a = m_2/y^b \in M_{xy}$ and ν is multiplied into M by $x^{max\{a,b\}}$ and $y^{max\{a,b\}}$. Therefore $M' \cong \ker(M_x \oplus M_y \to M_{xy})$.

Since both x and y are non-zerodivisors on M, we know $M \subseteq M_x, M_y \subseteq M_{xy}$. This means $M \hookrightarrow M_x \oplus M_y$ and the image of M is just the set of elements of the form $(m/1, m/1) \in M_x \oplus M_y$. Thus we have $M'/M \cong H^1_I(M)$ and will be done if we can show this is a finitely generated module.

We first show $H^1_{(x,y)}(M)$ is finitely generated when $M = R/P$ for some prime $P \in \mathrm{Ass}_R M$.

Since P kills $H^1_{(x,y)}(R/P)$, we can work over the domain $\bar{R} = R/P$. By hypothesis we know that $\bar{R}$ has an S_2-ification, S, so we have $0 \to \bar{R} \to S \to S/\bar{R} \to 0$ which induces

$$\cdots \to H^0_{(x,y)}(S/\bar{R}) \to H^1_{(x,y)}(\bar{R}) \to H^1_{(x,y)}(S) \to \cdots$$

Because $\mathrm{ht}(x,y)R/P = 2$ makes $\mathrm{depth}_{(x,y)} S = 2$, we know $H^1_{(x,y)}(S) = 0$. Therefore $H^1_{(x,y)}(\bar{R})$ is a quotient of $H^0_{(x,y)}(S/\bar{R})$ which is finitely generated, meaning $H^1_{(x,y)}(\bar{R})$ is a finitely generated module.

In the general case, take a filtration of M, $M_0 \subset M_1 \subset \cdots \subset M_n = M$, where each M_i is a torsion-free module over R/P_i for some $P_i \in \mathrm{Ass}_R M$. By induction on the length of the filtration, it is enough to show that we have $H^1_{(x,y)}(M_i)$ finitely generated for every i.

Because M_i is torsion-free over R/P_i, we have $0 \to M_i \to (R/P_i)^{d_i} \to N \to 0$ which gives:

$$\cdots \to H^0_{(x,y)}(N) \to H^1_{(x,y)}(M_i) \to (H^1_{(x,y)}(R/P_i))^{d_i} \to \cdots$$

Since we already know $H^1_{(x,y)}(R/P_i)$ is finitely generated, we must have $H^1_{(x,y)}(M_i)$ finitely generated as well. Therefore $H^1_{(x,y)}(M)$ is finitely generated as claimed. $\square$

Theorem III.4. *Let R be a ring which has an S_2-ification for every prime cyclic module. Then $Ass_R H^2_I(M)$ is finite whenever $Ass_R M \subseteq Ass_R R$ and $ht(IR/P) \geq 2$ for all $P \in Ass_R M$.*

Proof. Because $\mathrm{ht}(IR/P) \geq 1$ for all associated primes of M, we can pick $x \in I$ so that x is not in any associated prime of M, so x is a non-zerodivisor on M. Next pick $y \in I - xR$ which is not in any associated prime of M(We can do this because $\mathrm{ht}(IR/P) \geq 2$.) This makes y also a non-zerodivisor on M.

Let M' be defined as in Lemma III.2. Since x, y are non-zerodivisors on M so $M \hookrightarrow M_{xy}$, we get a short exact sequence $0 \to M \to M' \to M'/M \to 0$ which induces

$$\cdots \to H^1_I(M') \to H^1_I(M'/M) \to H^2_I(M) \to H^2_I(M') \to \cdots$$

By Lemma III.2, x, y is a possibly improper regular sequence on M', so $\mathrm{depth}_I M' \geq 2$. This means our long exact sequence above is actually

$$0 \to H^1_I(M'/M) \to H^2_I(M) \to H^2_I(M') \to \cdots$$

We know $\mathrm{Ass} H^1_I(M'/M)$ is finite by Proposition II.6 since M'/M is finitely generated. Also, because $\mathrm{depth}_I(M') = 2$, we have $\mathrm{Ass} H^2_I(M')$ is finite which implies that $\mathrm{Ass} H^2_I(M)$ is finite as well. $\square$

If $\dim(R) \leq 3$, Proposition II.7 and Corollary II.5 show that the only local cohomology module which could have infinitely many associated primes is $H^2_I(M)$, which gives our next Theorem as a corollary. First we require another lemma. We will call a module *skinny* if each of its quotients has only finitely many associated primes.

Lemma III.5. *Let R be a Noetherian ring, M any R-module. If M is finitely supported, then M is skinny. In particular this is true of modules with finitely many associated primes all of which are maximal in R.*

Proof. If $M_P = 0$ it is clearly impossible for any quotient, $\bar{M}$, of M to have $\bar{M}_P \neq 0$. Thus the support of any quotient of M is again finite, and since the associated primes of any module are contained in its support we are done. $\square$

Now we can proceed to our second main result.

Theorem III.6. *If R has an S_2-ification for every prime cyclic module, dim$(R) = 3$ and ht$(IR/P) \geq 2$ for every $P \in Ass_R M$ then $Ass_R H_I^i(M)$ is always finite.*

Proof. We are only concerned about the associated primes of $H_I^2(M)$, so if we can reduce to the case where M has pure dimension 3 we will have Ass$M \subseteq$ Ass$R$, and Theorem III.4 will finish this off.

If $\dim(M) < 3$, then $H_I^2(M)$ has only finitely many associated primes by Corollary II.5 since we may work mod the annihilator of M making $\dim(R) \leq 2$. Thus we may assume $\dim(M) = 3$.

We next reduce to the case where M has pure dimension. In doing so we may choose M so that any proper quotient, $\bar{M}$, of M has Ass$H_I^2(\bar{M})$ finite.

Let $N \subset M$ be any nonzero submodule. Our goal is to show that $\dim(N) = 3$.

The usual short exact sequence $0 \to N \to M \to M/N \to 0$, induces

$$\cdots \to H_I^2(N) \to H_I^2(M) \to H_I^2(M/N) \to H_I^3(N) \to \cdots$$

From this it is immediately clear that if $\dim(N) < 2$ we are done, because then $H_I^2(N) = H_I^3(N) = 0$. This makes $H_I^2(M) \cong H_I^2(M/N)$ implying that $H_I^2(M)$ has only finitely many associated primes.

Suppose $\dim(N) = 2$. Here we only have $H_I^3(N) = 0$, which gives us the short exact sequence

$$0 \to \mathrm{im}(H_I^2(N)) \to H_I^2(M) \to H_I^2(M/N) \to 0.$$

Any homomorphic image of $H_I^2(N)$ is a quotient by some submodule, so what we really need to know is that $H_I^2(N)$ is skinny. We may think over $R/\mathrm{Ann}(N)$, so we may assume that $\dim(R) = 2$. By Corollary II.5, we know that $H_I^2(N)$ has only finitely many associated primes all of which must be of height 2 and thus maximal. Lemma III.6 now shows that $H_I^2(N)$ is skinny, so $\mathrm{Ass}_R\mathrm{im}(H_I^2(N))$ is finite. By hypothesis $H_I^2(M/N)$ has only finitely many associated primes, so $\mathrm{Ass}H_I^2(M)$ is finite as well. This means that we can reduce to the case where M has pure dimension 3, where by Theorem III.4 we are done. □

3.2 Ideals of height less than two

In the last section we proved some results when $\mathrm{ht}(I) \geq 2$, but here we look at the case where $\mathrm{ht}(I) \leq 1$. We show that if R is a local ring of dimension 4 the question of whether $\mathrm{Ass}H_I^i(M)$ is finite reduces to a very concrete situation. Some parts of the proof follow the proof of [Mar01, Proposition 2.8]. There is a counter-example in dimension 4 with an ideal of height 1, see [SS04, Remark 4.2], but there the ring is neither local nor graded suitably for localization.

We will call a ring R *standard* if $R = V[[X_1, X_2, X_3, Y]]/(f)$ or $k[[X_1, \ldots, X_4, Y]]/(f)$ for some complete DVR V or field k, and f is a monic polynomial in Y with constant term divisible by X_1.

Proposition III.7. *If for every standard ring, R, we have $\mathrm{Ass}_R H^2_{(X_1,Y)}(G)$ finite for every finitely generated faithful module, G, of pure dimension 4, then $\mathrm{Ass}_R H_I^i(M)$ is finite for every four-dimensional local ring (R, m) and every finitely generated R-module M.*

Proof. Let (R, m) be local of dimension 4. Pick any ideal $I \subset R$ and a finitely generated R-module M. First note that by [Mar01] Proposition 2.8 we are done if $\mathrm{ht}(I) \geq 2$, so we can focus on $\mathrm{ht}(I) \leq 1$. Since we can kill $H_I^0(M) \subseteq M$ without affecting the problem, we may assume that $\mathrm{depth}_I M \geq 1$. Thus $\mathrm{ht}(I) = 1$. By Proposition II.7 and Corollary II.6, our only possible problem with $H_I^i(M)$ is when $i = 2$. Also note that we may assume that $R = \hat{R}$ is complete. We will first reduce to the case where M has dimension 4.

We may choose a minimal counter-example with respect to quotients, i.e. assume that $H^2_I(M/N)$ has only finitely many associated primes for any nonzero submodule $N \subseteq M$. If $\dim(M) \leq 3$ we may work modulo $\mathrm{Ann}(M)$ where we are done by [Mar01, Corollary 2.7], so $\dim(M) = 4$. Next we reduce to the case of pure dimension.

Let N be any nonzero submodule of M, so $0 \to N \to M \to M/N \to 0$ induces

$$\cdots \to H^2_I(N) \to H^2_I(M) \to H^2_I(M/N) \to H^3_I(N) \to \cdots$$

As in the proof of Theorem III.5, it is immediately clear that N cannot have dimension 0 or 1 since $\dim(N) < 2$ forces $H^2_I(N) = H^3_I(N) = 0$. This makes $H^2_I(M) \cong H^2_I(M/N)$ implying that $H^2_I(M)$ has only finitely many associated primes.

Now suppose $\dim(N) = 2$. As before we have $H^3_I(N) = 0$, which gives us

$$0 \to \mathrm{im}(H^2_I(N)) \to H^2_I(M) \to H^2_I(M/N) \to 0,$$

and what we really need to know is that $H^2_I(N)$ is skinny. We may think over $R/\mathrm{Ann}(N)$, so we may assume that $\dim(R) = 2$. By Corollary II.5, we know that $H^2_I(N)$ has only finitely many associated primes all of which must be of height 2 and thus maximal. This makes $H^2_I(N)$ is skinny by Lemma III.6, so $\mathrm{Ass}_R \mathrm{im}(H^2_I(N))$ is finite. Since $H^2_I(M/N)$ has only finitely many associated primes by hypothesis, so does $H^2_I(M)$.

If $\dim(N) = 3$, we still get

$$0 \to \mathrm{im}(H^2_I(N)) \hookrightarrow H^2_I(M) \to H^2_I(M/N)$$

exact. When thinking of $H^2_I(N)$, we may work over $R/\mathrm{Ann}(N)$ which is a local ring of dimension 3. The support of $H^2_I(N)$ is a finite set of height 2 primes and possibly the image of m, so $H^2_I(N)$ is skinny which means that $\mathrm{Ass}H^2_I(M)$ is finite by our previous argument.

We may therefore assume that M has pure dimension 4, and by killing the annihilator of M in R we may also assume M is faithful. We now reduce to the case where I is the intersection of primes with height 0 or 1.

Since $H^i_I(M) = H^i_{\sqrt{I}}(M)$, we may take $I = \bigcap_i P_i$. Let $A = \bigcap_{\mathrm{ht}(P)\leq 1} P$ and $B = \bigcap_{\mathrm{ht}(Q)\geq 2} Q$, so that $I = A \cap B$. This gives us a long exact sequence

$$\cdots \to H^1_{A+B}(M) \to H^2_I(M) \to H^2_A(M) \oplus H^2_B(M) \to H^2_{A+B}(M) \to \cdots$$

The support of $H^i_{A+B}(M)$ is contained in $V(A+B)$. But since $\mathrm{ht}(A+B) \geq 3$, we know any prime $Q \in V(A+B)$ is either one of finitely many minimal primes, ie $\mathrm{ht}(Q) = 3$, or $Q = m$. This means $H^i_{A+B}(M)$ is finitely supported and thus skinny. Since $\mathrm{ht}(B) \geq 2$, [Mar01] Proposition 2.8 shows that $\mathrm{Ass}H^2_B(M)$ is finite. Thus to show that $H^2_I(M)$ has finitely many associated primes it is enough to see $\mathrm{Ass}H^2_A(M)$ is finite, so in considering this problem we may assume that every associated prime of I has height at most 1.

Because M is faithful we have $R \hookrightarrow M^{\oplus n}$ for some n. Since $\mathrm{depth}_I M \geq 1$ which implies $\mathrm{depth}_I R \geq 1$, we know that I cannot be contained in any associated prime of R. Thus I is of pure height 1, i.e. $I = \cap_{i=1}^r P_i$ with $\mathrm{ht}(P_i) = 1$.

Since R is complete, we can find a regular subring A of R with $A \subseteq R$ module-finite. Pick some non-zerodivisor, $x \in I$. Let $I_0 = \sqrt{(xA)^e}$ be the radical of the expansion of xA to R. Clearly $I_0 \subseteq I$, so if we take a primary decomposition of I_0 it will be $I_0 = (\cap_{i=1}^r P_i) \cap (\cap_{j=1}^s Q_j)$ for some additional primes Q_j. We can pick another non-zerodivisor, $y \in I$ with $y \notin Q_j$ for all j. (In particular this means that $y \notin xR$.) Let $J = \sqrt{I_0 + \sqrt{(yA)^e}} = (\cap_{i=1}^r P_i) \cap (\cap_{j=1}^t \widetilde{Q_j})$. Since all these associated primes of J must contain x and y, and the only height one primes containing x are associated to I_0 while $y \notin Q_j$ for every j, we must have $\mathrm{ht}(\widetilde{Q_j}) \geq 2$ for all j. By repeating the proof that we have no primes of height 2 in the primary decomposition of $\sqrt{I}$ with $A = \cap_{i=1}^{r} P_i = I$ and $B = \cap_{j=1}^{t} \widetilde{Q_j}$, we can see that if $\mathrm{Ass}_R H_I^2(M)$ is infinite, so is $\mathrm{Ass}_R H_J^2(M)$. Thinking of M as a module over the ring $A[x,y] \subseteq R$, we can see that primes in $\mathrm{Ass}_R H_{(x,y)}^2(M)$ lie over primes of $\mathrm{Ass}_{A[x,y]} H_{(x,y)}^2(M)$. The extension of rings is module-finite so if $\mathrm{Ass}_R H_{(x,y)}^2(M)$ is infinite, $\mathrm{Ass}_{A[x,y]} H_{(x,y)}^2(M)$ is infinite as well.

As $x \in I \subseteq R$ is a non-zerodivisor, we can extend $x = x_1$ to a full system of parameters, $\underline{\mathrm{x}} = x_1, x_2, x_3, x_4$ for R. Letting V be a coefficient ring (or possibly field) for R, we can take $A = V[[\underline{\mathrm{x}}]]$. We can view $A[x,y] = A[y]$ as a quotient of the ring of formal power series $V[[X_1, X_2, X_3, X_4, Y]]$ by letting $X_i \mapsto x_i$ and $Y \mapsto y$. The kernel of this map is clearly a height one prime which, as R, and hence A, has pure dimension, is a prime of pure height one. Because the formal power series ring is a UFD, our prime is principal which means $A[y] \cong V[[X_1, X_2, X_3, X_4, Y]]/(f)$. Because the $X's$ form a system of parameters, some power of Y is in the ideal generated by the $X's$, which forces f to have a term which is some unit of V or k times a power of Y. By the Weierstrauss preperation theorem, we can replace f by an associate which is monic in Y. Since X_1, Y cannot be a regular sequence, we must have some multiple of Y in the ideal generated by X_1. There is only one relation in $A[y]$, so the constant term of our polynomial must be divisible by X_1. Therefore $f = Y^d - f_{d-1}Y^{d-1} - \cdots - f_d$ where $x_1 | f_d$.

We have now reduced the problem to asking whether $H_{(X_1,Y)}^2(G)$ has finitely many associated primes where G is a faithful module of pure dimension 4 over a standard ring as claimed.

$\square$

CHAPTER IV

Adjoining Indeterminates

Our fourth chapter explores two cases where the finiteness of the set of associated primes of local cohomology of the ring itself is inherited by the ring of polynomials or the ring of formal power series. Here we look only at the local cohomology of the ring itself since this work relies on similar results known for regular local rings.

In the first section we use some techniques from algebraic geometry to give a relationship between our polynomial or power series ring and some regular rings. The second section gives a result about polynomial rings in mixed characteristic p.

4.1 Rings with nice blowups

In this section we use the fact that our base ring has an easily controlled blowup to show that adjoining any finite set of interminates, either as polynomials or power series, preserves the finiteness of $\mathrm{Ass}_R H_I^i(R)$. As a corollary we get that $\mathrm{Ass}_R H_I^i(R)$ is finite for polynomial and power series rings over a normal domain of dimension two or three with an isolated singularity.

The proofs of our two main theorems are both done in the polynomial case, but the same proofs work for power series. In both cases, we rely on the following lemma.

Lemma IV.1. *Let R be a domain whose resolution of singularities, Y, is the blowup of R along I. If $\mathrm{depth}_I R \geq 2$ then $H^0(Y, \mathcal{O}_Y) \cong R$.*

Proof. Let $X = \operatorname{Spec}(R)$, $U = \operatorname{Spec}(R) - V(I)$ and $\widetilde{U}$ be its preimage in Y. We have a long exact sequence

$$0 \to H^0_I(R) \to H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X) \to H^1_I(R) \to \cdots$$

and $\operatorname{depth}_I R \geq 2$ forces $H^0_I(R) = H^1_I(R) = 0$, so we have

$$R = H^0(X, \mathcal{O}_X) \cong H^0(U, \mathcal{O}_X).$$

The map from Y to X is an isomorphism when we restrict to $\widetilde{U} \to U$, so we know that

$$H^0(U, \mathcal{O}_X) \cong H^0(\widetilde{U}, \mathcal{O}_Y).$$

But $\mathcal{O}_Y$ has no torsion since Y is the blowup of a domain, which means $H^0(Y, \mathcal{O}_Y) = H^0(\widetilde{U}, \mathcal{O}_Y)$.

Combining these equalities we see that $R \cong H^0(Y, \mathcal{O}_Y)$ as claimed. □

The next two theorems show that if our base ring has a blowup covered by a small number of affine patches then the local cohomology of a polynomial or power series extension has only finitely many associated primes.

Theorem IV.2. *Let A be a domain finitely generated as an algebra over a field, k, of characteristic 0 and $R = A[t_1, \ldots, t_n]$ or $A[[t_1, \ldots, t_n]]$. If A has a resolution of singularities, Y_0, which is the blowup of A along an ideal of depth at least two, with an affine open cover by only U_1 and U_2 where $H^1(Y_0, \mathcal{O}_{Y_0})$ has finite length over A, then $Ass_R H^i_I(R)$ is finite for any i and any ideal $I \subset R$.*

Proof. Take generators so that $I = (f_1, \ldots, f_n)$.

Let $A_0^{\bullet\bullet}$ be the double complex formed by tensoring the complex used to compute the sheaf cohomology of $\mathcal{O}_Y$,

$$0 \to S_1 \oplus S_2 \to S_{12} \to 0,$$

with the complex used to compute local cohomology of R,

$$0 \to R \to \oplus R_{f_i} \to \oplus R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_n} \to 0.$$

Thus

$$A_0^{i0} = \bigoplus (S_1 \oplus S_2)_{f_{k_1} \cdots f_{k_i}}$$

and

$$A_0^{i1} = \bigoplus (S_{12})_{f_{k_1} \cdots f_{k_i}},$$

so that $A_0^{\bullet\bullet}$ is the complex given below.

$$\begin{tikzcd}
& 0 & 0 & \\
0 \arrow[r] & (S_1 \oplus S_2)_{f_1 \cdots f_n} \arrow[u] \arrow[r] & (S_{12})_{f_1 \cdots f_n} \arrow[u] \arrow[r] & 0 \\
& {} \arrow[u] & {} \arrow[u] & \\
0 \arrow[r] & \bigoplus (S_1 \oplus S_2)_{f_k} \arrow[u] \arrow[r] & \bigoplus (S_{12})_{f_k} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & S_1 \oplus S_2 \arrow[u] \arrow[r] & S_{12} \arrow[u] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

We can filter this complex by subcomplexes $A_0^{\bullet\bullet}\langle \ell \rangle$ which are simply $A_0^{\bullet\bullet}$ with the first ℓ rows replaced by zeros. Let E_0 be the associated graded complex with respect to this filtration, which is just the direct sum of the rows. As described in 2.2, E_1 is just the total complex of $A_1^{\bullet\bullet}$ where

$$A_1^{i0} = \bigoplus H^0(Y, \mathcal{O}_Y)_{f_{k_1} \cdots f_{k_i}}$$

and

$$A_1^{i1} = \bigoplus H^1(Y, \mathcal{O}_Y)_{f_{k_1} \cdots f_{k_i}}.$$

Thus $A_1^{\bullet\bullet}$ is the double complex below where all horizontal maps are 0 and all vertical maps are those induced by the vertical maps of $A_0^{\bullet\bullet}$.

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & H^0(Y, \mathcal{O}_Y)_{f_1 \cdots f_n} \arrow[u] \arrow[r, "0"] & H^1(Y, \mathcal{O}_Y)_{f_1 \cdots f_n} \arrow[u] \arrow[r] & 0 \\
 & {} \arrow[u] & {} \arrow[u] & \\
0 \arrow[r] & \bigoplus H^0(Y, \mathcal{O}_Y)_{f_k} \arrow[u] \arrow[r, "0"] & \bigoplus H^1(Y, \mathcal{O}_Y)_{f_k} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & H^0(Y, \mathcal{O}_Y) \arrow[u] \arrow[r, "0"] & H^1(Y, \mathcal{O}_Y) \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $d^1 : E_1^{p,q} \to E_1^{p+1,q}$, so we are simply taking cohomology along each column. This means that E_2 is the total complex of $A_2^{\bullet\bullet}$ where

$$A_2^{i0} = H_I^i(H^0(Y, \mathcal{O}_Y))$$

and

$$A_2^{i1} = H_I^i(H^1(Y, \mathcal{O}_Y)).$$

Since $d^2 : E_2^{p,q} \to E_2^{p+2,q-1}$, or equivalently $d^2 : A_2^{i,j} \to A_2^{i+2,j-1}$, we are just mapping up two rows and over one column. With that in mind, we show the relevant three rows of $A_2^{\bullet\bullet}$ below.

$$\begin{tikzcd}
 & \uparrow & \uparrow & \\
0 \arrow[r] & H_I^{i+2}(H^0(Y,\mathcal{O}_Y)) \arrow[r] & H_I^{i+2}(H^1(Y,\mathcal{O}_Y)) \arrow[r] & 0 \\
0 \arrow[r] & H_I^{i+1}(H^0(Y,\mathcal{O}_Y)) \arrow[r] \arrow[u] & H_I^{i+1}(H^1(Y,\mathcal{O}_Y)) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & H_I^{i}(H^0(Y,\mathcal{O}_Y)) \arrow[r] \arrow[u] & H_I^{i}(H^1(Y,\mathcal{O}_Y)) \arrow[r] \arrow[u] & 0 \\
 & \arrow[u] & \arrow[u] &
\end{tikzcd}$$

We know that $d^2(A_2^{i,0}) = 0$ for any i since there are no nonzero rows to the left of the 0th row. Thus the only nontrivial instance of this map is $d^2 : A_2^{i,1} \to A_2^{i+2,0}$. Its kernel is $E_3^{i,1}$ since $d^2(A_2^{i-2,2}) \subseteq A_2^{i,1}$ is clearly zero. Similarly, the cokernel of this map is $E_3^{i+2,0} = A_2^{i+2,0}/d^2(A_2^{i,1})$ since all of $d^2 \equiv 0$.

It is worth noting that since d^r corresponds to going up r rows and left $r-1$ columns, we will have $d^r \equiv 0$ for all $r \geq 3$ meaning that $E_3 = E_\infty$.

Piecing all this together, for every i we get an exact sequence

$$0 \to E_\infty^{i,1} \to A_2^{i,1} \to A_2^{i+2,0} \to E_\infty^{i+2,0} \to 0.$$

However, letting $\mathcal{T}^\bullet$ denote the total complex E_∞, we know that $\mathcal{T}^i = E_\infty^{i,0} \oplus E_\infty^{i-1,1}$. This is because any element, z, of $E_\infty^{i,0}$ comes from an element of the row homology of our original double complex which is killed by the column map in $A_1^{\bullet\bullet}$. Thus we can form an element $(z,0)$ in $\mathcal{T}^i$ since elements of T^i are pairs $(x,y) \in A_2^{i,0} \oplus A_2^{i-1,1}$ where x maps to zero in $A_2^{i+1,0}$ and x and y map to the same thing in $A_2^{i,1}$. Since the cokernel is then clearly $E_\infty^{i-1,1}$, we get a short exact sequence

$$0 \to E_\infty^{i,0} \to \mathcal{T}^i \to E_\infty^{i-1,1} \to 0.$$

Combining these two exact sequences we get a long exact sequence

$$\cdots \to A_2^{i-1,0} \to \mathcal{T}^{i-1} \to A_2^{i-2,1} \to A_2^{i,0} \to \mathcal{T}^{i} \to A_2^{i-1,1} \to \cdots$$

which in our case is the long exact sequence

$$\cdots \to \mathcal{T}^{i-1} \to H_I^{i-2}(H^1(Y,\mathcal{O}_Y)) \to H_I^{i}(H^0(Y,\mathcal{O}_Y)) \to \mathcal{T}^{i} \to \cdots$$

We are interested in the associated primes of $H_I^i(H^0(Y,\mathcal{O}_Y))$, since by Lemma IV.1 we know $H_I^i(H^0(Y,\mathcal{O}_Y)) \cong H_I^i(R)$.

To understand the behavior of $\mathcal{T}^i$, we go back to our original double complex $B_0^{\bullet\bullet} := A_0^{\bullet\bullet}$ and filter the other way, i.e. so that $B_0^{\bullet\bullet}\langle k\rangle$ is just $B_0^{\bullet\bullet}$ with the first k columns replaced by zeros. As before, the associated graded complex, E_0, with respect to this filtration is the direct sum of the columns. This makes E_1 the total complex of $B_1^{\bullet\bullet}$ where $B_1^{i0} = H_I^i(S_1 \oplus S_2)$ and $B_1^{i1} = H_I^i(S_{12})$. Thus $B_1^{\bullet\bullet}$ is the double complex below where the vertical maps are 0 and horizontal maps induced by the corresponding maps in $B_0^{\bullet\bullet}$.

$$\begin{tikzcd}
 & 0 & 0 & \\
0 \arrow[r] & H_I^n(S_1 \oplus S_2) \arrow[r] \arrow[u] & H_I^n(S_{12}) \arrow[r] \arrow[u] & 0 \\
 & {} \arrow[u, "0"] & {} \arrow[u, "0"] & \\
 & {} & {} & \\
0 \arrow[r] & H_I^1(S_1 \oplus S_2) \arrow[r] \arrow[u, "0"] & H_I^1(S_{12}) \arrow[r] \arrow[u, "0"] & 0 \\
0 \arrow[r] & H_I^0(S_1 \oplus S_2) \arrow[r] \arrow[u, "0"] & H_I^0(S_{12}) \arrow[r] \arrow[u, "0"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $d^1 : E_1^{p,q} \to E_1^{p,q+1}$, so to get E_2 we are taking cohomology along each row. The ith row is the Čech complex which computes cohomology of the sheaf $\mathcal{H}^i_I(\mathcal{O}_Y)$ with respect to the cover of Y by U_1 and U_2. Thus E_2 is the total complex of $B_2^{\bullet\bullet}$ where $B_2^{i0} = H^0(Y, \mathcal{H}^i_I(\mathcal{O}_Y))$ and $B_2^{i1} = H^1(Y, \mathcal{H}^i_I(\mathcal{O}_Y))$. Here the map is $d^2 : E_2^{p,q} \to E_2^{p-1,q+2}$, or down one row and right two columns. Since there are only two nonzero columns, this map is the zero map on all of B_2, which means that $E_\infty = E_2$.

Again letting $\mathcal{T}^\bullet$ be the total complex E_∞, we have $\mathcal{T}^i = B_2^{i,0} \oplus B_2^{i-1,1}$. Any element, z, of $B_2^{i,0}$ is a column cycle of the original complex, so it maps to zero in $B_2^{i+1,0}$, and is also a row cycle in the column homology, so there is an element, w, of $B_2^{i-1,0}$ which maps to the image of z in $B_2^{i,1}$. This means we can exactly match such z's with elements (z, w) of $\mathcal{T}^i$. As before, it is clear the cokernel of this injection is $B_2^{i-1,1}$ so we get

$$0 \to B_2^{i,0} \to \mathcal{T}^i \to B_2^{i-1,1} \to 0,$$

which in our case is just

$$0 \to H^0(Y, \mathcal{H}^i_I(\mathcal{O}_Y)) \to \mathcal{T}^i \to H^1(Y, \mathcal{H}^{i-1}_I(\mathcal{O}_Y)) \to 0.$$

Both $H^i_I(H^0(Y, \mathcal{O}_Y))$ and $H^0(Y, \mathcal{H}^i_I(\mathcal{O}_Y))$ map to $\mathcal{T}^i$; the first map coming from the long exact sequence involving $A_2^{i,j}$'s and the second, which is an injection, coming from one of the short exact sequences involving $B_2^{i,j}$'s.

Lemma IV.3. *The map, σ, from $H^i_I(H^0(Y, \mathcal{O}_Y))$ to $\mathcal{T}^i$ has $\sigma : H^i_I(H^0(Y, \mathcal{O}_Y)) \to H^0(Y, \mathcal{H}^i_I(\mathcal{O}_Y)) \subset \mathcal{T}^i$.*

Proof. The module $H^0(Y, \mathcal{H}^i_I(\mathcal{O}_Y))$ is the row cohomology of the column cohomology of our original complex. This means its elements are represented by elements $z \in A_0^{i,0}$ where $z \mapsto 0 \in A_0^{i+1,0}$ (here we look at the class of z modulo the image of $A_0^{i-1,0}$) so that z is an element of the column cohomology. We also have $[z] \mapsto \operatorname{im}(A_0^{i-1,1}) \subseteq A_0^{i,1}$ so it is in the row cohomology of the column cohomology. In the last step we don't need to worry about taking further equivalence classes because $A_0^{i,-1}$, and hence its image, is zero.

Thus elements of $H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ can be viewed as the classes $z + \operatorname{im}(A_0^{i-1,0})$ for elements $z \in A_0^{i,0}$ where $z \mapsto 0 \in A_0^{i+1,0}$ and $z \mapsto \operatorname{im}(A_0^{i-1,1}) \subset A_0^{i,1}$. These sit inside $\mathcal{T}^i$ as pairs $([z],[w])$ where $w \in A_0^{i-1,1}$ and z have the same image in $A_0^{i,1}$.

On the other hand, the module $H_I^i(H^0(Y, \mathcal{O}_Y))$ is column cohomology of the row cohomology of $A_0^{\bullet\bullet}$. Its elements can be thought of as coming from elements $z \in A_0^{i,0}$ for which $z \mapsto 0 \in A_0^{i,1}$ (here we need not take classes since $A_0^{i,-1} = 0$) and where $z \mapsto 0$ in the row cohomology of the $(i+1,0)$th spot. Since $A_0^{i+1,-1} = 0$, this just means $z \mapsto 0 \in A_0^{i+1,0}$. Here the equivalence classes are with respect to the image of the row cohomology at the $(i-1,0)$th spot inside the row cohomology at the $(i,0)$th spot, i.e. $\operatorname{im}(\ker(A_0^{i-1,0} \to A_0^{i-1,1}))$.

Thus elements of $H_I^i(H^0(Y, \mathcal{O}_Y))$ are just classes $z + \operatorname{im}(\ker(A_0^{i-1,0} \to A_0^{i-1,1}))$ with $z \in A_0^{i,0}$ chosen so that $z \mapsto 0 \in A_0^{i,1}$ and $z \mapsto 0 \in A_0^{i+1,0}$. These sit inside $\mathcal{T}^i$ as elements of the type $([z],0)$.

Since we clearly have $0 \in \operatorname{im}(A_0^{i-1,1}) \subseteq A_0^{i,1}$, every element which represents a class of $H_I^i(H^0(Y, \mathcal{O}_Y))$ is also representative of a class in $H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$. Furthermore, since

$$\operatorname{im}(\ker(A_0^{i-1,0} \to A_0^{i-1,1})) \subseteq \operatorname{im}(A_0^{i-1,0}) \subseteq A_0^{i,0}$$

the map is well-defined. Therefore

$$\sigma(H_I^i(H^0(Y, \mathcal{O}_Y))) \subseteq H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y)) \subseteq \mathcal{T}^i.$$

$\square$

Because of this lemma, we may replace $\mathcal{T}^i$ by $H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ in our long exact sequence to get the exact sequence

$$\mathcal{T}^{i-1} \to H_I^{i-2}(H^1(Y, \mathcal{O}_Y)) \to H_I^i(H^0(Y, \mathcal{O}_Y)) \to H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y)).$$

The last module, $H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$, is the cohomology of the sequence

$$0 \to H_I^i(S_1) \oplus H_I^i(S_2) \to H_I^i(S_12) \to 0$$

at the 0th spot. This means

$$H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y)) \subseteq H_I^i(S_1) \oplus H_I^i(S_2).$$

Since both S_1 and S_2 are regular, each is a finite direct sum of regular domains. Cohomology commutes with direct sums, and by II.8 (a) and (d) we know regular domains finitely generated over a field of characteristic 0 have only finitely many primes associated to any local cohomology module of the ring. Thus $\mathrm{Ass}_{S_j} H_I^i(S_j)$ is finite for both our regular rings. However, the associated primes of $H_I^i(S_j)$ over R will be restrictions of the associated primes over S_j so we know that $\mathrm{Ass}_R H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ is finite.

This means that we only need to control the associated primes of

$$\mathrm{im}(H_I^{i-2}(H^1(Y, \mathcal{O}_Y)) \subseteq H_I^i(H^0(Y, \mathcal{O}_Y))$$

and we will be done.

To do this we will use D-module methods. We take $D = k[t_1, \ldots, t_m, \partial_1, \ldots, \partial_m]$ where ∂_i is differentiation with respect to t_i. Clearly $k[t_1, \ldots, t_m]$ has an action on R, and we can extend this to an action of D on R by setting $\partial_i(a) = 0$ for all $a \in A$.

S_1 and S_2 are generated over R by finitely many fractions of elements from R, so we can define an action of D on them by letting

$$\partial_i\left(\frac{r}{s}\right) = \frac{s\partial_i(r) - r\partial_i(s)}{s^2}.$$

Since this D action extends to all localizations and is compatible with localization maps every module in $A_0^{\bullet\bullet}$ is a D-module and all its maps are D-module maps. In fact, all the modules and maps in both spectral sequences will have a D-module structure, which means that the image of $\mathcal{T}^{i-1}$ inside $H_I^{i-2}(H^1(Y, \mathcal{O}_Y))$ is a D-submodule.

Because $H^1(Y_0, \mathcal{O}_{Y_0})$ has finite length over A, when we view it as a module over k it becomes just a finite dimensional vector space, k^a, for some a. Thus

$$H^1(Y, \mathcal{O}_Y) \cong H^1(Y_0, \mathcal{O}_{Y_0}) \otimes_k k[t_1, \ldots, t_m]$$

is just a direct sum of a copies of $k[t_1, \ldots, t_m]$, which is holonomic by Property 2.2(a) and Remark 2.9 from [Lyu93]. Proposition II.8 (d) shows that $H_I^{i-2}(H^1(Y, \mathcal{O}_Y))$ is holonomic, and since $\mathrm{im}(\mathcal{T}^{i-1})$ is holonomic by II.8 (c)

$$\mathrm{im}(H_I^{i-2}(H^1(Y, \mathcal{O}_Y))) = H_I^{i-2}(H^1(Y, \mathcal{O}_Y))/\mathrm{im}(\mathcal{T}^{i-1})$$

is holonomic. Therefore, by II.8 (e) and (f), $\mathrm{im}(H_I^{i-2}(H^1(Y, \mathcal{O}_Y)))$ has finitely many associated primes both over $k[t_1, \ldots, t_m]$ and over R. This makes

$$\mathrm{Ass}_R H_I^i(H^0(Y, \mathcal{O}_Y)) = \mathrm{Ass}_R H_I^i(R)$$

finite and finishes the proof.

□

The proof of the case where Y is covered by three open sets is similar in spirit, but the fact that the spectral sequences converge one step later creates an extra level of complexity.

Theorem IV.4. *Let A is a domain finitely generated as an algebra over a field, k, of characteristic 0 and $R = A[t_1, \ldots, t_m]$ or $A[[t_1, \ldots, t_m]]$. If A has a resolution of singularities, Y_0, which is the blowup of A along an ideal of depth at least two, which* *has an open cover by only U_1, U_2, and U_3 where $H^1(Y_0, \mathcal{O}_{Y_0})$ and $H^2(Y_0, \mathcal{O}_{Y_0})$ have finite length over A, then $Ass_R H^i_I(R)$ is finite for any i and any ideal $I \subset R$.*

Proof. Again, choose generators so that $I = (f_1, \dots, f_n) \subset R$.

Let $A_0^{\bullet\bullet}$ be the double complex formed by tensoring the complex used to compute the sheaf cohomology of $\mathcal{O}_Y$,

$$0 \to S_1 \oplus S_2 \oplus S_3 \to S_{12} \oplus S_{13} \oplus S_{23} \to S_{123} \to 0,$$

with the complex used to compute local cohomology of R

$$0 \to R \to \oplus R_{f_i} \to \oplus R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_n} \to 0.$$

Thus we have

$$A_0^{i0} = \bigoplus (S_1 \oplus S_2 \oplus S_3)_{f_{k_1} \cdots f_{k_i}},$$

$$A_0^{i1} = \bigoplus (S_{12} \oplus S_{13} \oplus S_{23})_{f_{k_1} \cdots f_{k_i}},$$

and

$$A_0^{12} = \bigoplus (S_{123})_{f_{k_1} \cdots f_{k_i}}$$

so that $A_0^{\bullet\bullet}$ is the complex given below.

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & (S_1 \oplus S_2 \oplus S_3)_{f_1 \cdots f_n} \arrow[u] \arrow[r] & (S_{12} \oplus S_{13} \oplus S_{23})_{f_1 \cdots f_n} \arrow[u] \arrow[r] & (S_{123})_{f_1 \cdots f_n} \arrow[u] \arrow[r] & 0 \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & \\
0 \arrow[r] & \bigoplus (S_1 \oplus S_2 \oplus S_3)_{f_k} \arrow[u] \arrow[r] & \bigoplus (S_{12} \oplus S_{13} \oplus S_{23})_{f_k} \arrow[u] \arrow[r] & (S_{123})_{f_k} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & S_1 \oplus S_2 \oplus S_3 \arrow[u] \arrow[r] & S_{12} \oplus S_{13} \oplus S_{23} \arrow[u] \arrow[r] & S_{123} \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

As in the case with two patches, we first filter this complex by subcomplexes $A_0^{\bullet\bullet}\langle\ell\rangle$ which are simply $A_0^{\bullet\bullet}$ with the first ℓ rows replaced by zeros. Let E_0 be the associated graded complex with respect to this filtration so that E_1 is the total complex of $A_1^{\bullet\bullet}$ where

$$A_1^{i0} = \bigoplus H^0(Y, \mathcal{O}_Y)_{f_{k_1}\cdots f_{k_i}},$$

$$A_1^{i1} = \bigoplus H^1(Y, \mathcal{O}_Y)_{f_{k_1}\cdots f_{k_i}},$$

and

$$A_1^{i2} = \bigoplus H^2(Y, \mathcal{O}_Y)_{f_{k_1}\cdots f_{k_i}}.$$

Thus $A_1^{\bullet\bullet}$ is the double complex below where all horizontal maps are 0 and all vertical maps are those induced by the vertical maps of $A_0^{\bullet\bullet}$.

$$\begin{tikzcd}
& 0 & 0 & 0 & \\
0 \arrow[r] & H^0(Y, \mathcal{O}_Y)_{f_1\cdots f_n} \arrow[u] \arrow[r, "0"] & H^1(Y, \mathcal{O}_Y)_{f_1\cdots f_n} \arrow[u] \arrow[r] & H^2(Y, \mathcal{O}_Y)_{f_1\cdots f_n} \arrow[u] \arrow[r] & 0 \\
& {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & \\
0 \arrow[r] & \bigoplus H^0(Y, \mathcal{O}_Y)_{f_k} \arrow[u] \arrow[r, "0"] & \bigoplus H^1(Y, \mathcal{O}_Y)_{f_k} \arrow[u] \arrow[r] & \bigoplus H^2(Y, \mathcal{O}_Y)_{f_k} \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & H^0(Y, \mathcal{O}_Y) \arrow[u] \arrow[r, "0"] & H^1(Y, \mathcal{O}_Y) \arrow[u] \arrow[r] & H^1(Y, \mathcal{O}_Y) \arrow[u] \arrow[r] & 0 \\
& 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $d^1 : E_1^{p,q} \to E_1^{p+1,q}$, so we are simply taking cohomology along each column.

This means that E_2 is the total complex of $A_2^{\bullet\bullet}$ where

$$A_2^{i0} = H_I^i(H^0(Y, \mathcal{O}_Y)),$$

$$A_2^{i1} = H_I^i(H^1(Y, \mathcal{O}_Y)),$$

and

$$A_2^{i2} = H_I^i(H^2(Y, \mathcal{O}_Y)).$$

Since $d^2 : E_2^{p,q} \to E_2^{p+2,q-1}$, or equivalently $d^2 : A_2^{i,j} \to A_2^{i+2,j-1}$, we are just mapping up two rows and over one column. With that in mind, we show the relevant three rows of $A_2^{\bullet\bullet}$ below.

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H_I^{i+2}(H^0(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i+2}(H^1(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i+2}(H^2(Y,\mathcal{O}_Y)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H_I^{i+1}(H^0(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i+1}(H^1(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i+1}(H^2(Y,\mathcal{O}_Y)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H_I^{i}(H^0(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i}(H^1(Y,\mathcal{O}_Y)) & \longrightarrow & H_I^{i}(H^2(Y,\mathcal{O}_Y)) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

This means that E_3 is the total complex of $A_3^{\bullet\bullet}$ where

$$A_3^{ij} = \frac{\ker(d^2 : A_2^{ij} \to A_2^{i+2,j-1})}{d^2(A_2^{i-2,j+1})}.$$

Since we have only three nonzero columns, this gives us

$$A_3^{i0} = \frac{A_2^{i0}}{d^2(A_2^{i-2,1})} = \frac{H_I^i(H^0(Y,\mathcal{O}_Y))}{d^2(H_I^{i-2}(H^1(Y,\mathcal{O}_Y)))},$$

$$A_3^{i1} = \frac{\ker(d^2 : A_2^{i1} \to A_2^{i+2,0})}{d^2(A_2^{i-2,2})} = \frac{\ker(H_I^i(H^1(Y,\mathcal{O}_Y)) \to H_I^{i+2}(H^0(Y,\mathcal{O}_Y)))}{d^2(H_I^{i-2}(H^2(Y,\mathcal{O}_Y)))},$$

and

$$A_3^{i2} = \ker(d^2 : A_2^{i2} \to A_2^{i+2,1}) = \ker(H_I^i(H^2(Y, \mathcal{O}_Y)) \to H_I^{i+2}(H^1(Y, \mathcal{O}_Y))).$$

The map $d^3 : E_3^{p,q} \to E_3^{p+3,q-2}$ or $d^3 : A_3^{ij} \to A_3^{i+3,j-2}$ is just up three rows and left two columns. Since we only have three nonzero columns, this map is only nontrivial if $j = 2$ so $d^3 : A_3^{i2} \to A_3^{i+3,0}$. This means that

$$E_4^{i0} = \frac{A_3^{i0}}{d^3(A_3^{i-3,2})},$$

$$E_4^{i1} = A_3^{i1},$$

and

$$E_4^{i2} = \ker(d^3 : A_3^{i2} \to A_3^{i+3,0}).$$

Since d^r corresponds to going up r rows and left $r-1$ columns, we have $d^r \equiv 0$ for all $r \geq 4$ meaning that $E_4 = E_\infty$.

From this we can construct an exact sequence

$$0 \to E_\infty^{i-3,2} \to A_3^{i-3,2} \to A_3^{i,0} \to E_\infty^{i,0} \to 0$$

for every i where the middle map is d^3.

Letting $\mathcal{T}^\bullet$ denote the total complex E_∞, we know that $\mathcal{T}^i = E_\infty^{i,0} \oplus E_\infty^{i-1,1} \oplus E_\infty^{i-2,2}$. This gives us an exact sequence

$$\mathcal{T}^{i-1} \to A_3^{i-3,2} \to A_3^{i,0} \to \mathcal{T}^i$$

where the first map is projection onto $E_\infty^{i-3,2}$, the middle map is d^3 and the last map comes from the injection $E_\infty^{i,0} \hookrightarrow \mathcal{T}^i$.

In our case, this is actually

$$\mathcal{T}^{i-1} \to \ker(d^2 : H_I^{i-3}(H^2(Y, \mathcal{O}_Y)) \to H_I^{i-1}(H^1(Y, \mathcal{O}_Y))) \to \frac{H_I^i(H^0(Y, \mathcal{O}_Y))}{d^2(H_I^{i-2}(H^1(Y, \mathcal{O}_Y)))} \to \mathcal{T}^i$$

Our ring, R, satisfies the hypotheses of Lemma IV.1, so

$$H_I^i(H^0(Y, \mathcal{O}_Y))) \cong H_I^i(R)$$

which means we are interested in controlling the associated primes of the second to last module in our exact sequence.

As in the proof of Theorem IV.2, we let $D = k[t_1, \ldots, t_m, \partial_1, \ldots, \partial_m]$ where ∂_i is differentiation with respect to t_i. By letting $\partial_i(a) = 0$ for all $a \in A$, we have an action of D on R and hence on each S_i. This action extends to all localizations, so all modules in $A_3^{\bullet\bullet}$ are D-modules and all maps are D-module maps. Since $H^2(Y_0, \mathcal{O}_{Y_0})$ is finite length over A, it is a finite vector space over k. This makes $H^2(Y, \mathcal{O}_Y) \cong H^2(Y_0, \mathcal{O}_{Y_0}) \otimes_k k[t_1, \ldots, t_m]$ a direct sum of copies of $k[t_1, \ldots, t_m]$ which is a holonomic D-module. As before, by Proposition II.8 (a), we know $H_I^i(H^2(Y, \mathcal{O}_Y))$ is holonomic. This means $\ker(d^2 : H_I^{i-2}(H^2(Y, \mathcal{O}_Y)) \to H_I^i(H^1(Y, \mathcal{O}_Y)))$, and thus the image of $\mathcal{T}^{i-1}$ inside this kernel, is also holonomic. Therefore

$$\operatorname{im}(d^3) \subseteq \frac{H_I^i(H^0(Y, \mathcal{O}_Y))}{d^2(H_I^{i-2}(H^1(Y, \mathcal{O}_Y)))}$$

has a finite set of associated primes over $k[t_1, \ldots, t_m]$ and over R.

To control the image of $A_3^{i,0}$ inside $\mathcal{T}^i$, we go back to our original double complex $B_0^{\bullet\bullet} := A_0^{\bullet\bullet}$ and filter the other way, i.e. so that $B_0^{\bullet\bullet}\langle k \rangle$ is just $B_0^{\bullet\bullet}$ with the first k columns replaced by zeros. As before, the associated graded complex, E_0, with respect to this filtration is the direct sum of the columns. This makes E_1 the total complex of $B_1^{\bullet\bullet}$ where $B_1^{i0} = H_I^i(S_1 \oplus S_2 \oplus S_3)$, $B_1^{i1} = H_I^i(S_{12} \oplus S_{13} \oplus S_{23})$ and $B_1^{i2} = H_I^i(S_{123})$. Thus $B_1^{\bullet\bullet}$ is the double complex below where the vertical maps are 0 and horizontal maps induced by the corresponding maps in $B_0^{\bullet\bullet}$.

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & H_I^n(S_1 \oplus S_2 \oplus S_3) \arrow[r] \arrow[u] & H_I^n(S_{12} \oplus S_{13} \oplus S_{23}) \arrow[r] \arrow[u] & H_I^n(S_{123}) \arrow[r] \arrow[u] & 0 \\
 & {} \arrow[u, "0"] & {} \arrow[u, "0"] & {} \arrow[u, "0"] & \\
 & {} & {} & {} & \\
0 \arrow[r] & H_I^1(S_1 \oplus S_2 \oplus S_3) \arrow[r] \arrow[u, "0"] & H_I^1(S_{12} \oplus S_{13} \oplus S_{23}) \arrow[r] \arrow[u, "0"] & H_I^1(S_{123}) \arrow[r] \arrow[u, "0"] & 0 \\
0 \arrow[r] & H_I^0(S_1 \oplus S_2 \oplus S_3) \arrow[r] \arrow[u, "0"] & H_I^0(S_{12} \oplus S_{13} \oplus S_{23}) \arrow[r] \arrow[u, "0"] & H_I^0(S_{123}) \arrow[r] \arrow[u, "0"] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Here $d^1 : E_1^{p,q} \to E_1^{p,q+1}$, so to get E_2 we are taking cohomology along each row. The ith row is the Čech complex which computes cohomology of the sheaf $\mathcal{H}_I^i(\mathcal{O}_Y)$ with respect to the cover of Y by U_1, U_2 and U_3. Thus E_2 is the total complex of $B_2^{\bullet\bullet}$ where $B_2^{i0} = H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$, $B_2^{i1} = H^1(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ and $B_2^{i2} = H^2(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$. The differential here is $d^2 : E_2^{p,q} \to E_2^{p-1,q+2}$, or down one row and right two columns. Since there are only three nonzero columns, this map is the zero except for $d^2 : B_2^{i0} \to B_2^{i-1,2}$.

This means we have E_3 the total complex of $B_3^{\bullet\bullet}$ where

$$B_3^{i0} = \ker(B_2^{i0} \to B_2^{i-1,2}),$$

$$B_3^{i1} = B_2^{i,2}$$

and

$$B_3^{i2} = \frac{B_2^{i2}}{d^2(B_2^{i+1,0})}.$$

Because there are only three nonzero columns and d^3 maps down two rows and over three columns, $d^3 \equiv 0$ which means $E_\infty^{ij} = E_3^{ij}$. For each i this gives us a short exact sequence

$$0 \to E_\infty^{i0} \to B_2^{i0} \to B_2^{i-1,2} \to E_\infty^{i-1,2} \to 0$$

where the middle map is d^2.

Again letting $\mathcal{T}^\bullet$ be the total complex E_∞, we have $\mathcal{T}^i = B_3^{i,0} \oplus B_3^{i-1,1} \oplus B_3^{i-2,2}$. Putting this together with the exact sequence we from our other filtration of $A_0^{\bullet\bullet}$ we have

$$B_2^{i-1,1} \oplus B_2^{i-2,2} \to \mathcal{T}^i \to B_2^{i,0} \to B_2^{i-1,2}$$

which in our case is just

$$H^1(Y, \mathcal{H}_I^{i-1}(\mathcal{O}_Y)) \oplus H^2(Y, \mathcal{H}_I^{i-2}(\mathcal{O}_Y)) \to \mathcal{T}^i \to H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y)) \to H^2(Y, \mathcal{H}_I^{i-1}(\mathcal{O}_Y)).$$

Here the first map is the direct sum of the inclusion of $B_2^{i-1,1}$ into $\mathcal{T}^i$ and the map of $B_2^{i-2,2}$ onto $E_\infty^{i-1,2}$, the second is killing $E_\infty^{i-1,1} \oplus E_\infty^{i-2,2}$ inside $\mathcal{T}^i$ and mapping $E_\infty^{i,0}$ into $B_2^{i,0}$, while the last map is just d^2.

Notice that the image of $\mathcal{T}^i$ is a submodule of $H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ which is in turn contained in $H_I^i(S_1) \oplus H_I^i(S_2) \oplus H_I^i(S_3)$.

Since S_1, S_2 and S_3 are all regular, $\mathrm{Ass}_{S_j} H_I^i(S_j)$ is finite for each of our regular rings. Because the associated primes of $H_I^i(S_j)$ over R will be restrictions of the associated primes over S_j, we conclude that $\mathrm{Ass}_R H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ is finite. From this we see $\mathrm{Ass}_R \mathrm{im}(\mathcal{T}^i)$ is finite as well.

The image of A_3^{i0} in $\mathcal{T}^i$ also maps to $B_2^{i,0}$ and the kernel is the intersection of $\mathrm{im}(A_3^{i0})$ and the image in $\mathcal{T}^i$ of $B_0^{i-1,1} \oplus B_0^{i-2,2}$.

Lemma IV.5. *The image of $A_3^{i,0}$ in $\mathcal{T}^i$ has trivial intersection with the image of $B_2^{i-1,1} \oplus B_2^{i-2,2}$.*

Proof. The module $A_3^{i,0}$ comes from our filtration by setting rows equal to zero. Its elements can be thought of as coming from elements $z \in A_0^{i,0}$ for which $z \mapsto 0 \in A_0^{i,1}$ (here we need not take classes since $A_0^{i,-1} = 0$) and where $z \mapsto 0$ in the row cohomology of the $(i+1,0)$th spot. Since $A_0^{i+1,-1} = 0$, this just means $z \mapsto 0 \in A_0^{i+1,0}$. Here the equivalence classes are with respect to the image of the row cohomology at the $(i-1,0)$th spot inside the row cohomology at the $(i,0)$th spot, i.e. $\mathrm{im}(\ker(A_0^{i-1,0} \to A_0^{i-1,1}))$. This puts $[z] \in A_2^{i,0}$.

Since d^2 maps up two rows and left one column, it kills all of $A_2^{i,0}$. This means z represents an element of $A_3^{i,0}$ although we are now working mod $d^2(A_2^{i-2,1})$. Elements of $A_2^{i-2,1}$ are classes of elements in $A_0^{i-2,1}$ which map to zero in $A_0^{i-2,2}$ and map to $\mathrm{im}(A_0^{i-1,0}) \subseteq A_0^{i-1,1}$. The map d^2 works by taking such an element, w, to $\mathrm{im}(y) \in A_0^{i,0}$ where the element $y \in A_0^{i-1,0}$ has $\mathrm{im}(y) = \mathrm{im}(w) \in A_0^{i-1,1}$.

Thus $A_3^{i,0}$ is classes of elements from $A_0^{i,0}$ which map to zero in both $A_0^{i+1,0}$ and $A_0^{i,1}$ mod the image of any element of $A_0^{i-1,0}$ which maps to $\mathrm{im}(A_0^{i-2,1}) \subseteq A_0^{i-1,1}$.

Elements of $\mathcal{T}^i$ are classes of triples (z, w, v) in $A_0^{i,0} \oplus A_0^{i-1,1} \oplus A_0^{i-2,2}$ where $z \mapsto 0 \in A_0^{i+1,0}$, $\mathrm{im}(z) = \mathrm{im}(w) \in A_0^{i,1}$ and $\mathrm{im}(w) = \mathrm{im}(v) \in A_0^{i-1,2}$. Therefore our elements of $A_3^{i,0}$ map to $\mathcal{T}^i$ by $[z] \mapsto [(z, 0, 0)]$.

Similarly, $B_2^{i-2,2}$ comes from the filtration where we set columns equal to zero. This means its elements are represented by elements $v \in A_0^{i-2,2}$ where $v \mapsto 0 \in A_0^{i-1,2}$ (here we look at the class of v modulo the image of $A_0^{i-3,2}$) so that $[v]$ is an element of the column cohomology. Since $B_0^{i-2,3} = 0$, v automatically represents an element of the row cohomology of the column cohomology. However, we work mod the image of the column cohomology at the $(i-2,1)$th spot.

Thus elements of $B_2^{i-2,2}$ are classes of $v \in B_0^{i-2,2}$ where we work mod both $\mathrm{im}(B_0^{i-3,2})$ and $\mathrm{im}(\ker(B_0^{i-2,1} \to B_0^{i-1,1}))$ inside $B_0^{i-2,2}$. This means that our map

$B_2^{i-2,2} \to \mathcal{T}^i$ is by sending $[v] \mapsto [(0,0,v)]$.

Our last module, $B_2^{i-1,1}$, comes from the same filtration as $B_0^{i-1,1}$. This means it is classes of elements $w \in B_0^{i-1,1}$ where $w \mapsto 0 \in B_0^{i,1}$ working mod the image of $B_0^{i-2,1}$ in $B_0^{i-1,1}$. We also need that $w \mapsto \operatorname{im}(B_0^{i-2,2}) \subseteq B_0^{i-1,2}$, and we work mod $\operatorname{im}(\ker(B_0^{i-1,0} \to B_0^{i,0})) \subseteq B_0^{i-1,1}$.

Therefore elements of $B_2^{i-1,1}$ are classes of elements $w \in B_0^{i-1,1}$ where $w \mapsto 0 \in B_0^{i,1}$ and we have an element $v \in B_0^{i-2,2}$ with $\operatorname{im}(w) = \operatorname{im}(v) \in B_0^{i-1,2}$. Here we work modulo $\operatorname{im}(B_0^{i-2,1})$ and $\operatorname{im}(\ker(B_0^{i-1,0} \to B_0^{i,0}))$ inside $B_0^{i-1,1}$. Thus $B_0^{i-1,1} \to \mathcal{T}^i$ is just $[w] \mapsto [(0,w,v)]$.

Since $\operatorname{im}(A_3^{i,0})$ has entries only in the first component while the images of the other two modules have entries only in the last two components, it is clear that images intersect only in the zero triple.

$\square$

This lemma tells us that $\operatorname{im}(A_3^{i,0}) \subseteq \mathcal{T}^i$ actually injects into $\operatorname{im}(\mathcal{T}^i) \subseteq H^0(Y, \mathcal{H}_I^i(\mathcal{O}_Y))$ which means that $\operatorname{Ass}_R \operatorname{im}(A_3^{i,0})$ is finite. Therefore $\operatorname{Ass}_R A_3^{i,0}$ is finite as well.

Since we are really interested in $H_I^i(H^0(Y,\mathcal{O}_Y)) \cong H_I^i(R)$, and

$$A_3^{i,0} = \frac{H_I^i(H^0(Y,\mathcal{O}_Y))}{d^2(H_I^{i-2}(H^1(Y,\mathcal{O}_Y)))},$$

we consider the exact sequence

$$0 \to d^2(H_I^{i-2}(H^1(Y,\mathcal{O}_Y))) \to H_I^i(H^0(Y,\mathcal{O}_Y)) \to A_3^{i,0}.$$

We only need to show that $\operatorname{Ass}_R d^2(H_I^{i-2}(H^1(Y,\mathcal{O}_Y)))$ is finite and we are done.

Using D-modules as we did with $H^2(Y,\mathcal{O}_Y)$, we can show $H^1(Y_0,\mathcal{O}_{Y_0})$ is holonomic. This makes $H_I^{i-2}(H^1(Y,\mathcal{O}_Y))$ and hence its image under d^2 holonomic, which means $d^2(H_I^{i-2}(H^1(Y,\mathcal{O}_Y)))$ has finitely many associated primes over R.

Thus $\mathrm{Ass}_R H^i_I(R)$ is finite for any i and any $I \subset R$.

□

The following corollary is a special case of these two theorems whose assumptions are perhaps more familiar.

Corollary IV.6. *Let A be a two or three dimensional normal domain finitely generated as an algebra over a field of characteristic 0, and $R = A[t_1, \ldots, t_n]$ or $A[[t_1, \ldots, t_n]]$. If A has an isolated singularity, then $Ass_R H^i_I(R)$ is finite for any ideal in R.*

Proof. Let $m \subset A$ be the maximal ideal, which defines the non-singular locus of A. Since $\dim(A) \leq 3$, we know m is generated, up to radical, by at most three elements. Let Y_0 be the blow-up of A along m. It is clear that Y_0 is covered by at most three affine patches corresponding to the generators of m, and also that $\mathrm{depth}_m R \geq 2$ since $\mathrm{ht}(m) \geq 2$ and R is normal. Finally, we know that the higher cohomology of the structure sheaf of any desingularization of A will be finitely generated A-modules supported only on the singular locus of A. Since $\mathrm{Sing}(A) = \{m\}$, this means all higher cohomology of Y_0 is killed by some power of m and hence is of finite length over A. Theorems IV.2 and IV.4 now imply $\mathrm{Ass}_R H^i_I(R)$ is finite.

□

The only part of the proofs of Theorems IV.2 and IV.4 which uses the fact that we are in characteristic 0 is the D-module theory. In equal characteristic $p > 0$, Lyubeznik has successfully used his theory of "F-modules" to control the local cohomology of regular local rings in a way analogous to his use of D-modules in characteristic 0. I feel it should be possible to use those F-module techniques to adapt these proofs to the characteristic p case.

4.2 Over unramified rings of mixed characteristic

In this section we work with rings of mixed characteristic $p > 0$, i.e. where $\mathrm{char}(R) = 0$ while the maximal ideals each contain some prime number $p > 0$. We can ask whether, if a maximal ideal m contains p, we have $p \in m^2$. Here we will study rings where this does not happen, called *unramified* rings. This result shows that adjoining variables to an unramified regular local ring of mixed characteristic preserves the property that $\mathrm{Ass}_R(H^i_I(R))$ is finite for every i. The initial part of the proof follows the first half of Lyubeznik's proof of Theorem 3.2 in [Lyu97].

Theorem IV.7. *Let A be an unramified regular local ring of mixed characteristic $p > 0$. If $R = A[x_1, \ldots, x_n]$ is a polynomial ring over A for some n, then $Ass_R H^i_I(R)$ is finite for every integer i and ideal $I \subset R$.*

Proof. Let $R' = A[T_0, \ldots, T_n]$ be another polynomial ring over A in the new variables $T_0, \ldots, T_n$. We get an A-algebra map $f : R \to R'_{T_0}$ by $x_i \mapsto T_i/T_0$. Set W to be the multiplicative system $R' - P$ where P is the maximal ideal $m_A R' + (T_0, \ldots, T_n)$. There is a natural localization map $\ell : R'_{T_0} \to W^{-1}(R'_{T_0}) = (W^{-1}R')_{T_0}$. To simplify notation, we will let $B = (W^{-1}R')_{T_0}$ and $S = W^{-1}R'$.

By composition, we have a map $h : R \to B$ by $h = \ell \circ f$. Clearly h is R-flat, since it is the composition of two localization maps. We want to see that it is faithfully flat.

Pick any nonzero R-module, M. If we grade R'_{T_0} over $\mathbb{Z}$ by setting $\deg(T_i) = 1$ and $\deg(a) = 0$ for all $a \in A$, then $R'_{T_0} \otimes_R M = \bigoplus_{j \in \mathbb{Z}} T_0^j M$ is a $\mathbb{Z}$-graded R'_{T_0}-module where $\deg(u) = 0$ for every $u \in M$. Every element of our multiplicative system W is a polynomial in $T_0, \ldots, T_n$ whose constant term is a unit in A. Therefore it cannot kill any homogeneous element of $\bigoplus_{j \in \mathbb{Z}} T_0^j M$, so localizing at W we see

$W^{-1}(R'_{T_0}) \otimes_R M = B \otimes_R M$ is nonzero. Thus B is faithfully flat over R.

Since B is faithfully flat over R, if we have $\mathrm{Ass}_R H^i_I(R)$ infinite then by tensoring with B we will have $\mathrm{Ass}_B H^i_{IB}(B)$ infinite as well. But $B = S_{T_0}$, so every associated prime of $H^i_{IB}(B)$ corresponds to an associated prime of $H^i_{IB\cap S}(S)$ which does not contain T_0.

Thus $\mathrm{Ass}_R H^i_I(R)$ infinite forces $\mathrm{Ass}_S H^i_{IB\cap S}(S)$ to be infinite as well. However S is again an unramified regular local ring, so by [Lyu00b, Theorem 1], we know $\mathrm{Ass}_S H^j_J(S)$ is always finite, and therefore $\mathrm{Ass}_R H^i_I(R)$ must be finite.

$\square$

This same proof works for any class of rings where the finiteness of the associated primes of local cohomology is preserved by adjoining variables and then localizing at an ideal which contains all the variables plus a maximal ideal of the base ring.

CHAPTER V

Calm Extensions

In this chapter, we investigate properties of the special class of ring extensions defined below.

Definition 2. *We call an extension $R \to S$* calm *if for every $P \in Spec(R)$ we have a finite set of* attached primes, *$\mathfrak{a}(P) \subseteq Spec(S)$, so that for every finitely generated R-module M we have*

$$Ass_S S \otimes M \subseteq \bigcup_{P \in Ass_R M} \mathfrak{a}(P).$$

We are interested in such extensions because if $R \to S$ is calm then whenever an R-module, M, has only finitely many associated primes, so does its extension $S \otimes M$.

Our first section shows that calm extensions have many nice properties. The second gives some classes of rings which have only calm extensions, while the third investigates when module-finite extensions are calm. The fourth section shows that the results from sections two and three cannot be extended much farther by exhibiting examples of extensions which are not calm.

5.1 Properties of calm extensions

This section lists some of the basic properties of calm extensions as well as a few criteria for when an extension is calm.

The definition of a calm extension is motivated by the desire to generalize the following theorem about flat extensions.

Theorem V.1. *Let $R \to S$ be a flat extension of rings, and M any R module. Then*

$$Ass_S S \otimes M = \bigcup_{P \in Ass_R M} Ass_S(S \otimes R/P).$$

This means that all flat extensions are automatically calm. In fact $R \to S$ is calm whenever the non-flat locus of S over R is finite, i.e. whenever S_P is flat over R_P for all but finitely many primes $P \in \mathrm{Spec}(R)$.

These next theorems show that calm extensions persist under some basic algebraic operations.

Theorem V.2. *If $R \to S$ is calm and $I \subset R$ is any ideal, then $R/I \to S/IS$ is calm.*

Proof. Every prime $\bar{P} \in \mathrm{Spec}(R/I)$ corresponds to a prime $P \in \mathrm{Spec}(R)$ which contains I. Similarly primes $\bar{Q}$ of S/IS correspond to primes Q of S which contain IS.

Since $R \to S$ is calm, let $\mathfrak{a}(P) \subseteq \mathrm{Spec}(S)$ be the set of attached primes for $P \in \mathrm{Spec}(R)$. Given $\bar{P} \in \mathrm{Spec}(R/I)$ corresponding to $P \in \mathrm{Spec}(R)$, let $\bar{\mathfrak{a}}(\bar{P})$ be the set of primes in S/IS which correspond to the primes of S in $\mathfrak{a}(P)$.

Any module, M, over R/I can be viewed as a module over R with $I \subseteq \mathrm{Ann}_R M$ so $(S/IS) \otimes_{R/I} M \cong S \otimes_R M$. Since I kills M, and thus any associated prime of $S \otimes_R M$ must contain IS, it is clear that

$$\mathrm{Ass}_S S \otimes_R M \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P)$$

immediately implies

$$\mathrm{Ass}_{S/IS}(S/IS) \otimes_{R/I} M \subseteq \bigcup_{\bar{P} \in \mathrm{Ass}_{R/I} M} \bar{\mathfrak{a}}(\bar{P}).$$

$\square$

Theorem V.3. *If $R \to S$ is calm and $W \subset R$ is any multiplicative system, then $W^{-1}R \to W^{-1}S$ is calm.*

Proof. Because the associated primes of a module after localization at W are just the original associated primes which avoid W, it is clear that we can find sets of attached primes which work for any $W^{-1}R$ module of the form $W^{-1}M$ for some R-module M.

Let N be any module finitely generated over the localized ring. We can represent N as the cokernel of a finite matrix with entries in $W^{-1}R$. This means we can find a single element, w, so that wN is the cokernel of a matrix with entries in R, meaning wN is an R-module. This makes $N = W^{-1}(wN)$ so we are done. $\square$

Theorem V.4. *If $R \to S$ and $S \to T$ are calm, then the composition $R \to T$ is calm.*

Proof. Since $R \to S$ and $S \to T$ are calm, we have sets $\mathfrak{a}(P) \subseteq \mathrm{Spec}(S)$ for each $P \in \mathrm{Spec}(R)$ and $\mathfrak{b}(Q) \subseteq \mathrm{Spec}(T)$ for each $Q \in \mathrm{Spec}(S)$ as in Definition 2. Let

$$\mathfrak{c}(P) = \bigcup_{Q \in \mathfrak{a}(P)} \mathfrak{b}(Q).$$

Then if M is any R-module we have

$$\mathrm{Ass}_T T \otimes_R M = \mathrm{Ass}_T T \otimes_S (S \otimes_R M) \subseteq \bigcup_{Q \in \mathrm{Ass}_S S \otimes_R M} \mathfrak{b}(Q)$$

But

$$\mathrm{Ass}_S S \otimes_R M \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P),$$

so we have

$$\bigcup_{Q \in \mathrm{Ass}_S S \otimes_R M} \mathfrak{b}(Q) \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \left(\bigcup_{Q \in \mathfrak{a}(P)} \mathfrak{b}(Q) \right) = \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{c}(P)$$

and $R \to T$ is calm.

□

The next theorems give two ways to check if an extension is calm. The second gives a partial converse to Theorem V.3, and says calmness can be checked locally on affine open covers of $\mathrm{Spec}(R)$.

Theorem V.5. *If R' is faithfully flat over R and $R' \to R' \otimes_R S$ is calm, then $R \to S$ is calm.*

Proof. Let $S' = R' \otimes_R S$. Because $R \to R'$ is flat, we know

$$\mathrm{Ass}_{R'} R' \otimes_R M = \bigcup_{P \in \mathrm{Ass}_R M} \mathrm{Ass}_{R'} R'/PR'.$$

Since $R' \to S'$ is calm, we have a set $\mathfrak{b}(P') \subseteq \mathrm{Spec}(S')$ for each $P' \in \mathrm{Spec}(R')$ so

$$\mathrm{Ass}_{S'} S' \otimes_{R'} (R' \otimes_R M) \subseteq \bigcup_{P' \in \mathrm{Ass}_{R'} R' \otimes_R M} \mathfrak{b}(P') = \bigcup_{P \in \mathrm{Ass}_R M} \left(\bigcup_{P' \in \mathrm{Ass}_{R'} R'/PR'} \mathfrak{b}(P') \right),$$

where $S' \otimes_{R'} (R' \otimes_R M) \cong S' \otimes_R M$. Now let

$$\mathfrak{a}(P) = \bigcup_{P' \in \mathrm{Ass}_{R'} R'/PR'} \{Q \in \mathrm{Spec}(S) | Q = Q' \cap S \text{ for some } Q' \in \mathfrak{b}(P')\}.$$

Then it is clear that

$$\{Q \in \mathrm{Spec}(S) | Q = Q' \cap S \text{ for some } Q' \in \mathrm{Ass}_{S'} S' \otimes_R M\} \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P).$$

Because $S \to S'$ is flat (by flat base change) and $S' \otimes_S (S \otimes_R M) = S' \otimes_R M$, we know that

$$\mathrm{Ass}_{S'} S' \otimes_R M = \bigcup_{Q \in \mathrm{Ass}_S S \otimes_R M} \mathrm{Ass}_{S'} S'/QS'.$$

Flatness also tells us that for each $Q \in \mathrm{Ass}_S S \otimes_R M$ there is some minimal prime of QS' which contracts to Q in S. Thus we have

$$\mathrm{Ass}_S S \otimes_R M \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P)$$

so $R \to S$ is calm. □

Theorem V.6. *Let $f_1, \ldots, f_n \in R$ generate the unit ideal. If $R_{f_i} \to S_{f_i}$ is calm for every i then $R \to S$ is calm.*

Proof. First note that the associated primes of a module over R_{f_i} correspond to its associated primes over R which do not contain f_i and similarly for S. Since $R_{f_i} \to S_{f_i}$ is calm for each i, we get sets $\mathfrak{b}_i(P)$ for each $P \in \mathrm{Spec}(R_{f_i})$ so that

$$\mathrm{Ass}_{S_{f_i}} S_{f_i} \otimes_{R_{f_i}} N \subseteq \bigcup_{P \in \mathrm{Ass}_{R_{f_i}} N} \mathfrak{b}_i(P)$$

for every R_{f_i}-module N. Now set

$$\mathfrak{a}(P) = \bigcup_{f_i \notin P} \{Q \in \mathrm{Spec}(S) | QS_{f_i} \in \mathfrak{b}_i(PR_{f_i})\}.$$

Let M be any R-module, and pick $Q \in \mathrm{Ass}_S S \otimes_R M$. There is some i for which $f_i \notin Q$, so

$$QR_{f_i} \in \mathrm{Ass}_{R_{f_i}} (S \otimes_R M)_{f_i} = \mathrm{Ass}_{R_{f_i}} S_{f_i} \otimes_{R_{f_i}} M_{f_i}.$$

Thus $QR_{f_i} \in \mathfrak{b}_i(PR_{f_i})$ for some $P \in \mathrm{Ass}_R M$ which means $Q \in \mathfrak{a}(P)$. Thus

$$\mathrm{Ass}_S S \otimes_R M \subseteq \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P)$$

and we are done. $\square$

5.2 Serene rings

In this section we investigate rings which have only calm extensions.

Definition 3. *We will call a ring R* serene *if $R \to S$ is calm for any ring S.*

Theorem V.7. *If R_1 and R_2 are serene, then $R = R_1 \times R_2$ is serene.*

Proof. Pick any map $R \to S$ and let M be any R-module. We know M has the form $M_1 \times M_2$ where M_1 is an R_1-module and M_2 is an R_2-module. This decomposition of M also gives a decomposition $\mathrm{Ass}_R M = \mathrm{Ass}_R M_1 \cup \mathrm{Ass}_R M_2$.

From the map $R \to S$ we get maps $R_1 \to S$ and $R_2 \to S$. Since both R_1 and R_2 are serene, we have sets $\mathfrak{b}(P)$ and $\mathfrak{c}(Q)$ in $\operatorname{Spec}(S)$ so that

$$\operatorname{Ass}_S S \otimes_{R_1} M_1 \subseteq \bigcup_{P \in \operatorname{Ass}_{R_1} M_1} \mathfrak{b}(P) \text{ and}$$

$$\operatorname{Ass}_S S \otimes_{R_2} M_2 \subseteq \bigcup_{Q \in \operatorname{Ass}_{R_2} M_2} \mathfrak{c}(Q).$$

Primes of R have the form $P \times R_2$ for $P \in \operatorname{Spec}(R_1)$ or $R_1 \times Q$ for $Q \in \operatorname{Spec}(R_2)$. Set $\mathfrak{a}(P \times R_2) = \mathfrak{b}(P)$ and $\mathfrak{a}(R_1 \times Q) = \mathfrak{c}(Q)$. Since all associated primes of M of the first type come from M_1 and all of the second type from M_2, we are done. $\square$

The next two results in this section give classes of rings which are serene.

Theorem V.8. *Dedekind domains are serene.*

Proof. Let R be a Dedekind domain and $R \to S$ a ring homomorphism. We can take $\mathfrak{a}(0) = \operatorname{Ass}_S S$, and $\mathfrak{a}(p) = \cup_m \operatorname{Ass}_S S/p^m S$ for each nonzero prime element $p \in R$. I claim the latter is a finite set of primes as follows: Filter each $S/p^m S$ by modules of the form $p^n S/p^{n+1} S$ so that

$$\bigcup_m \operatorname{Ass}_S S/p^m S \subseteq \bigcup_n \operatorname{Ass}_S p^n S/p^{n+1} S.$$

There is a system of surjections

$$S/pS \twoheadrightarrow pS/p^2 S \twoheadrightarrow p^2 S/p^3 S \twoheadrightarrow \cdots \twoheadrightarrow p^n S/p^{n+1} S \twoheadrightarrow \cdots$$

where each map is multiplication by p. Look at the kernel, Q_n, of the composition map $S/pS \twoheadrightarrow p^n S/p^{n+1} S$. Since any element killed by mapping n steps will be zero after any further mapping, we have

$$Q_1 \subseteq Q_2 \subseteq \cdots \subseteq Q_n \subseteq \cdots \subset S/pS.$$

This chain must stabilize since S/pS is Noetherian, and thus the modules $p^nS/p^{n+1}S \cong (S/pS)/Q_n$ stabilize as well making our union of assassinators really a finite union. As $\mathrm{Ass}_S p^nS/p^{n+1}S$ is finite for every n, this means that $\cup_m \mathrm{Ass}_S S/p^mS$ is finite as well.

Now let M be any finitely generated R-module. Because R is a Dedekind domain, we can decompose M as

$$M \cong R^{\oplus h} \oplus I \oplus R/P_i^{a_i} \oplus \cdots \oplus R/P_n^{a_n}$$

for some ideal $I \subseteq R$ and nonzero primes $P_1, \ldots, P_n \in \mathrm{Spec}(R)$. When we tensor with S we get

$$S \otimes M \cong S^{\oplus h} \oplus IS \oplus S/P_1^{a_1}S \oplus \cdots \oplus S/P_n^{a_n}S.$$

Since $\mathrm{Ass}_R R/P_i^m = \{P_i\}$, it is clear that $\mathrm{Ass}_R M = \{P_1, \ldots, P_n\}$ plus $\{(0)\}$ if $h > 0$. This makes it clear that

$$\mathrm{Ass}_S S \otimes M = \bigcup_{P \in \mathrm{Ass}_R M} \mathfrak{a}(P)$$

as required. □

Theorem V.9. *Regular rings of dimension 1 are serene.*

Proof. Since every regular ring of dimension 1 is a finite product of Dedekind domains, we are immediately done by combining Theorems V.7 and V.8.

□

5.3 Module-finite extensions

In this section we investigate which module-finite extensions are calm. For small rings, we are able to relax the conditions which made a ring serene and still have all module-finite extensions be calm.

Theorem V.10. *If $R \to S$ is module-finite,* $\dim(R) \leq 1$*, R is reduced and the singular locus of R is closed then $R \to S$ is calm.*

Proof. If $\dim(R) = 0$ this is trivial, because $\dim(R) = 0$ and $R \to S$ module-finite means that $\dim(S) = 0$. Since $\mathrm{Spec}(S)$ is finite, we can make $R \to S$ calm by letting $\mathfrak{a}(P) = \mathrm{Spec}(S)$ for every prime of R.

If $\dim(R) = 1$, we first tackle the domain case. If R is a domain with closed singular locus, we can find some nonzero element $a \in R$ so that R_a is regular.

Since R_a is regular of dimension 1, Theorem V.9 makes $R_a \to S_a$ calm. This means we need only worry about finding sets of attached primes for primes of R that contain a.

Since $\mathrm{ht}(aR) = 1$, these are the minimal primes of aR and hence there are only finitely many. Because $R \to S$ is module-finite, there are only finitely many primes of S lying over each of these primes in R which contain a. This means we can attach all these primes to every prime of R which contains a. Since every prime of S that contains a must lie over a prime of R which also contains a, we are done.

If R is not a domain but merely reduced, we will use induction on the number of minimal primes. If R has only one minimal prime it is a domain and we are done. If not, let $a_1, \ldots, a_n$ be a minimal set of elements of R which contains a generating set for every minimal prime. Since R_{a_i} has fewer minimal primes than R (because we have lost all primes which had a_i as a generator), the map $R_{a_i} \to S_{a_i}$ is calm by hypothesis. Thus we need only find attached sets of primes for all primes $Q \in \mathrm{Spec}(R)$ which contain every minimal prime of R.

The sum of all the minimal primes (which is contained in each of these Q's) has height 1, so the Q's are really its minimal primes. This means there are only finitely many of them, so by the argument used in the domain case we can just attach all primes of S lying over all of them to each Q. Thus $R \to S$ is calm. $\square$

Theorem V.11. *If R is local, reduced, and $\dim(R) = 2$, then $R \to S$ module-finite implies that $R \to S$ is calm.*

Proof. The flat locus of S over R is just the free locus, which is open in the module-finite case. Let $I \subset R$ define the non-flat locus. If $\mathrm{ht}(I) = 0$ and we localize at a minimal prime containing I we get a field. Thus $\mathrm{ht}(I) \geq 1$. The primes of S containing IS lie over the finitely many minimal primes of I and possibly the maximal ideal of R meaning the non-flat locus is finite. By our remark at the beginning of Section 5.1, this means $R \to S$ is calm. $\square$

5.4 Counter-examples

In this section we give several examples of extensions that are not calm even though some are module-finite. They show that the results of the last two sections cannot be pushed much farther.

The first example shows that rings of dimension 0 that are not reduced need not be serene.

Proposition V.12. *The ring $R = k[x, y]/(x^2, xy, y^2)$ is not serene.*

Proof. To begin, we will describe the structure of any finitely generated R module, M. Let $m = (x, y)$. First we can divide M into it's submodule mM and quotient M/mM. Since $m^2 = 0$, both are k-vector spaces and we can view M as a k-vector space. Here M splits up as $M = V \oplus W$ where V is the submodule mM and W is its vector space complement.

The actions of x and y on M are then the actions of two matrices, A and B, over k where $A, B : W \to V$ and both kill V. Choosing bases $v_1, \ldots, v_\ell$ and $w_1, \ldots, w_h$

for V and W respectively, we then have

$$\mathcal{A} = \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{B} = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

which map $V \oplus W$ to itself, and clearly satisfy the requirements that $\mathcal{A}^2$, $\mathcal{AB}$, $\mathcal{BA}$, and $\mathcal{B}^2$ are zero. Thinking of M in this way, we can map a free module of rank $\ell + h$ onto M by mapping each free module generator onto the v_i's and w_j's. From this we get the following representation of M:

$$0 \to N(A,B) \to R^\ell \oplus R^h \to M \to 0$$

where

$$N(A,B) = \mathrm{Span}_i\{xv_i, yv_i, xw_i - a_{i1}v_1 - \cdots - a_{ik}v_k, yw_i - b_{i1}v_1 - \cdots - b_{ik}v_k\}$$

and the a_{ij} and b_{ij} are the nonzero entries of the matrices $\mathcal{A}$ and $\mathcal{B}$. Thus to describe any finitely generated R-module it is only necessary to choose a pair of matrices A and B which correspond to the module

$$M \cong \frac{R^\ell \oplus R^h}{N(A,B)}.$$

Similarly an R-algebra, S, is simply a k-algebra with two elements $x, y \in S$ satisfying $x^2 = xy = y^2 = 0$.

From the short exact sequence $0 \to V \to M \to W \to 0$ we get $\cdots \to S \otimes V \to S \otimes M \to S \otimes W \to 0$, which means

$$\mathrm{Ass}_S \frac{S \otimes V}{\ker(S \otimes V \to S \otimes M)} \subseteq \mathrm{Ass}_S S \otimes M \subseteq \mathrm{Ass}_S S \otimes W \bigcup \mathrm{Ass}_S \frac{S \otimes V}{\ker(S \otimes V \to S \otimes M)}.$$

Since $W = M/V$,

$$S \otimes W \cong \frac{S \otimes (R^{\ell+h}/N(A,B))}{\mathrm{Span}_i\{v_i\}} \cong S^h/\mathrm{Span}_i\{xw_i, yw_i\} \cong (S/mS)^h$$

so we would like to have $\mathrm{Ass}_S S/mS \subseteq \mathfrak{a}(m)$. Since this set is finite, that is not a problem.

We can think of M as $R^{\ell+h}/N(A,B)$, so $S \otimes M \cong S^{\ell+h}/N(A,B)S$. As with $S \otimes W$, we can see $S \otimes V \cong \mathrm{Span}_i\{v_i\}S/\mathrm{Span}_i\{xv_i, yv_i\}$. This means that the kernel of the map $S \otimes V \to S \otimes M$, is precisely the nontrivial (meaning not over R) S-linear combinations of the $\alpha_i = xw_i - a_{i1}v_1 - \cdots - a_{ik}v_k$ and $\beta_i = yw_i - b_{i1}v_1 - \cdots - b_{ik}v_k$ which are in the span of the v_i's. Thus we are really looking for pairs of elements $s, t \in S$ where $sx + ty = 0$. (Even for flat maps this will include the pairs $(x,0)$, $(y,0)$, $(0,x)$ and $(0,y)$.) Each such pair gives a set of elements $(s,t) \cdot (\alpha_1, \beta_1), \ldots (s,t) \cdot (\alpha_h, \beta_h)$ in the kernel. This means $S \otimes V$ is isomorphic to $(S/mS)^k$ modulo all the relations described above.

For a fixed algebra S, we can take a finite basis $(s_1, t_i) \ldots, (s_d, t_d)$ for the relations on x and y, but we can have any module, M, which means literally any choice of the a's and b's. This means that our set of primes attached to m will have to include the set

$$\mathrm{Ass}_S \frac{(S/mS)^\ell}{\{(s_j, t_j) \cdot (\alpha_i, \beta_i)\}}$$

for every choice of ℓ, α and β.

We will conclude the proof by exhibiting an example, S, where this set of primes is infinite, meaning we cannot construct any suitable finite $\mathfrak{a}(m)$ showing R is not serene.

Let $S = k[x,y,u,v]/(x^2, xy, y^2, ux + vy)$, so our only new relation on x and y in S is (u,v). If we let $\ell = 1$ and consider the set above, we are looking at

$$\mathrm{Ass}_S \frac{S/mS}{\{(u,v) \cdot (a,b)\}} = \mathrm{Ass}_{k[u,v]} k[u,v]/(au + bv)$$

over all choices of a and b in k. But now observe that $(u + bv) \subset k[u,v]$ is prime for every $b \in k$ and furthermore that each such prime is distinct for different choices of b. This means $R \to S$ is not calm and therefore R is not serene. $\square$

The next example shows that rings of dimension 1 which are not reduced can have module-finite extensions which are not calm.

Proposition V.13. *The ring $R = k[x,y]/(y^2)$ has a module-finite R-algebra, S, where $R \to S$ is not calm.*

Proof. Given an R-module, M, we can think of M by thinking of M/yM and yM. Since $y^2 = 0$, both these module are killed by y making them effectively modules over $R/(y) = k[x]$. This ring is a PID, so we can decompose as follows

$$yM \cong k[x]^n \oplus k[x]/(f_1^{\alpha_1}) \oplus \cdots \oplus k[x]/(f_a^{\alpha_a}),$$

$$M/yM \cong k[x]^m \oplus k[x]/(g_1^{\beta_1}) \oplus \cdots \oplus k[x]/(g_b^{\beta_b})$$

where the f_i's and g_i's are prime elements of $k[x]$.

This means that $\mathrm{Ass}_R(M) \subseteq \{(y)\} \cup \{(y, f_i)\} \cup \{(y, g_i)\}$ so we will be looking for the sets of attached primes of S for these primes of R. (Note that $\mathrm{Spec}(R) = \{(y)\} \cup \{(y, f(x)) | (f) \in \mathrm{Spec}(k[x]\}$.)

We label the generators of these modules by choosing free bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$ for $k[x]^n$ and $k[x]^m$ respectively and letting w_i (respectively z_i) be a generator of $k[x]/(f_i^{\alpha_i})$ (respectively $k[x]/(g_i^{\beta_i})$) over $k[x]$. These two bases for yM and M/yM together give a generating set for M over $k[x]$.

To describe the R-module structure of M, we must now understand the action of y. Certainly any generator which came from yM is killed by y. Any generator for the part of M which maps to M/yM will be mapped to some $k[x]$-linear combination of the generators for yM. This means we can think of y as a matrix with entries in

$k[x]$. If we list the generators of M as u's, w's, v's, z's, then this matrix has the form

$$A = \begin{bmatrix} 0 & 0 & A^1 & A^2 \\ 0 & 0 & A^3 & A^4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We need to understand the associated primes of $S \otimes M$ for various module-finite R-algebras S. From the short exact sequence $0 \to yM \to M \to M/yM \to 0$ we get

$$\cdots \to yM \otimes S \to M \otimes S \to (M/yM) \otimes S \to 0.$$

From this we see that

$$\begin{aligned} \mathrm{Ass}_S \frac{S \otimes yM}{\ker(yM \otimes S \to M \otimes S)} &\subseteq \mathrm{Ass}_S M \otimes S \\ &\subseteq \mathrm{Ass}_S M/yM \otimes S \cup \mathrm{Ass}_S \frac{S \otimes yM}{\ker(yM \otimes S \to M \otimes S)} \end{aligned}$$

which means we need to understand these two additional modules and their associated primes over S.

From our previous decomposition of

$$M/yM \cong (R/yR)^m \oplus R/(y, g_1^{\beta_1}) \oplus \cdots \oplus R/(y, g_b^{\beta_b})$$

we see that

$$S \otimes (M/yM) \cong (S/yS)^m \oplus S/(y, g_1^{\beta_1})S \oplus \cdots \oplus S/(y, g_b^{\beta_b})S.$$

We can take care of these associated primes by including $\mathrm{Ass}_S S/yS$ in $\mathfrak{a}((y))$ and $\bigcup_t \mathrm{Ass}_S S/(y, g_i^t)S$ in $\mathfrak{a}((y, g_i))$. (This latter set is finite by the argument in the proof of Theorem V.8.)

To get at the kernel module, we will use our previously selected generating set for M along with the matrix A giving the action of y on M.

Mapping a free module onto M by sending each generator to one of our generators for M, we get that $M \cong R^{n+a+m+b}/\{\text{relations}\}$ where the set of relations is the span of things of the form

$$w_i f_i^{\alpha_i}, z_i g_i^{\beta_i}, yu_i, yw_i, yv_i - \gamma_i, yz_i - \delta_i \text{ where}$$

$$\gamma_i := a_{1i}^1 u_1 + \cdots + a_{ni}^1 u_n + a_{1i}^3 w_1 + \cdots + a_{ai}^3 w_a, \text{and}$$

$$\delta_i := a_{1i}^2 u_1 + \cdots + a_{ni}^2 u_n + a_{1i}^4 w_1 + \cdots + a_{ai}^4 w_a.$$

From this point of view, yM is the R-span of the u_i's and w_i's modulo the span of the relations yu_i, yw_i and $w_i f_i^{\alpha_i}$. Tensoring with S, we see that $S \otimes yM$ is has the same form, where the spans are taken over S instead of R. From this interpretation it is clear that the kernel of the map $S \otimes yM \to S \otimes M$ is the set of all S-linear (but not R-linear) combinations of the $yv_i - \gamma_i$'s, $yz_i - \delta_i$'s and $z_i g_i^{\beta_i}$'s which are contained in the S-span of the u_i's and w_i's. Putting $yv_i - \gamma_i$ into the span of the u_i's and w_i's means finding elements s of S which kill y since there are no other relations involving v_i. Putting $yz_i - \delta_i$ into the span of the u_i's and w_i's means finding elements t and t' of S so that $ty - t' g_i^{\beta_i} = 0$. (Putting $z_i g_i^{\beta_i}$ into yM means finding s' which kills $g_i^{\beta_i}$ so these do not give any new relations on the u_i's and v_i's.) From such s and t, t' we get kernel elements $s\gamma_i$ and $t\delta_i$.

We are assuming here that S is fixed, so there will be a finite set of generators for the s's and t's. However, as M is allowed to vary over all R-modules, n, m, a, b and the γ_i and δ_i (really the entries of A) are not fixed in any way.

To exhibit our counterexample, we will pick n and a along with a ring S so that various choices of γ_i will yield an infinite set of distinct associated primes. It will be clear that all the primes produced belong in the set $\mathfrak{a}((y))$ meaning that the map $R \to S$ cannot be calm.

Let $S = k[x, y, h]/(h^2, y^2, hy)$. This ring is generated over R by 1 and h. Here $\mathrm{Ann}(y) = (y, h)$ so for our set of elements s we will take (h). Consider modules where $n = 1$ and $a = 0$ so our matrix A has only one entry f which we can take to be any element of $k[x]$. We will let f run through the set of $x - c$'s where $c \in k$. The module whose associated primes we are considering is then

$$(S/yS)/(hf) \cong (k[x, h]/(h^2))/(h(x - c)) \cong k[x, h]/(h^2, h(x - c)).$$

Since $\sqrt{(h^2, h(x - c))} = (h, h(x - c)) = (h) \cap (h, x - c)$, we see that $(h, x - c)$ is an associated prime. Since such primes are distinct for different values of c, this means over an infinite field the set is infinite. Since the only associated prime of this M is (y), we must include all these primes in $\mathfrak{a}(y)$ which is clearly impossible. Thus the extension cannot be calm.

$\square$

Our next examples show that even a module-finite extension of domains need not be calm if our ring has dimension bigger than one.

Proposition V.14. *Let $R = k[x^2, x^3, y]$ and $S = k[x, y]$. The obvious injection $R \hookrightarrow S$ is not calm.*

Proof.

$$\text{Let } M_\lambda = \frac{Re_1 + Re_2}{(x^3e_2, x^2e_1, x^3e_1, (y - \lambda)e_1 - x^2e_2)}$$

for some $\lambda \in k$. This gives us a short exact sequence

$$0 \to Re_1 \to M_\lambda \to Re_2/(x^3e_2, x^2e_2) \to 0.$$

Clearly the last module in the sequence, M_λ/Re_1, is isomorphic to $R/(x^2, x^3)$. To understand the first module, Re_1, we need to compute $\mathrm{Ann}_R(e_1)$. It is clear that

$(x^2, x^3) \subset \mathrm{Ann}_R(e_1)$, so any other generator will have the form $g(y) \in k[y]$. Suppose that we have $g(y)e_1 = 0 \in M_\lambda$, i.e.

$$g(y)e_1 = r_1x^2e_1 + r_2x^3e_1 + r_3x^3e_2 + r_4((y-\lambda)e_1 - x^2e_2)$$

for some $r_1, r_2, r_3, r_4 \in R$. Since the left hand side has no e_2 term we must have $r_3x^3 - r_4x^2 = 0$, but all relations on x^2 and x^3 in R are spanned by $x^2(x^3) - x^3(x^2)$ and $x^3(x^3) - x^4(x^2)$. Thus $r_3 = ax^2 + bx^3$ and $r_4 = ax^3 + bx^4$ for some $a, b \in R$. Putting this back into the equation above we get

$$g(y)e_1 = r_1x^2e_1 + r_2x^3e_1 + (ax^3 + bx^4)(y-\lambda)e_1.$$

Since every term of the right hand side is divisible by x, this should mean $x|g(y)$ which is a contradiction unless $g(y) = 0$. Thus $\mathrm{Ann}_R(e_1) = (x^2, x^3)$, and $Re_1 \cong R/(x^2, x^3)$.

Putting this together with our short exact sequence for M_λ we see that $\mathrm{Ass}_R M_\lambda = \{(x^2, x^3)\}$ which does not depend at all on our choice of λ. This means that if our extension is calm the modules $S \otimes M_\lambda$ can have only finitely many associated primes as they will all have to live in $\mathfrak{a}((x^2, x^3))$.

Tensoring our exact sequence with S we get

$$\cdots \to S \otimes Re_1 \to S \otimes M_\lambda \to S \otimes Re_2/(x^3e_2, x^2e_2) \to 0, \text{ which means}$$

$$\mathrm{Ass}_S \frac{Se_1}{\ker(Se_1 \to S \otimes M_\lambda)} \subseteq \mathrm{Ass}_S S \otimes M_\lambda.$$

The kernel of the map from Se_1 to $S \otimes M_\lambda$ consists of all relations on e_1 which exist in $S \otimes M_\lambda$, i.e. all relations on e_1 in the span over S (but not R) of x^2e_1, x^3e_1, x^3e_2 and $(y-\lambda)e_1 - x^2e_2$. Since x^2 and x^3 already kill e_1 over R we discard anything in $(x^2, x^3)S = x^2S$. We do have $x((y-\lambda)e_1 - x^2e_2) + x^3e_2 = x(y-\lambda)e_1$ in the S-span of our relations, so $x(y-\lambda)e_1$ is in the kernel, and it is clear that this is the whole kernel. This means that $\ker(Se_1 \to S \otimes M_\lambda) = x(y-\lambda)e_1 S \subset Se_1$. Since $Se_1 \cong S/(x^2)$, this gives

$$\mathrm{Ass}_S S/(x^2, x(y-\lambda)) \subseteq \mathrm{Ass}_S S \otimes M_\lambda.$$

But

$$(x^2, x(y-\lambda)) = (x) \cap (x, y-\lambda)^2, \text{ so}$$

$$\mathrm{Ass}_S S/(x^2, x(y-\lambda)) = \{(x), (x, y-\lambda)\}.$$

Since the various $(x, y-\lambda)$'s are distinct for distinct values of λ, if k is infinite this gives a contradiction to the necessary finiteness of the attached sets of primes. (This is reasonable since the primes collected by varying λ are precisely the primes of S which lie over the non-flat locus of S in R.) □

Proposition V.15. *Let $R = k[x^3, x^4, y]$ and $S = k[x^2, x^3, y]$. $R \hookrightarrow S$ is module-finite but not calm.*

Proof. The obstacle to flatness here is the relation $x^3(x^3) = x^2(x^4) \in S$ but this cannot hold in R as $x^2 \notin R$. Also there are infinitely many maximal ideals of R which contain the two elements x^3 and x^4 which are involved in this relation. We will make use of that to construct our class of modules.

$$\text{Let } M_\lambda = \frac{Re_1 + Re_2}{(x^3 e_2, x^3 e_1, x^4 e_1, (y-\lambda)e_1 - x^4 e_2)}$$

for some $\lambda \in k$. This gives us a short exact sequence

$$0 \to Re_1 \to M_\lambda \to Re_2/(x^3 e_2, x^4 e_2) \to 0.$$

Clearly the last module in the sequence, M_λ/Re_1, is isomorphic to $R/(x^3, x^4)$. To understand the first module, Re_1, we need to compute $\mathrm{Ann}_R(e_1)$. It is clear that

$(x^3, x^4) \subset \mathrm{Ann}_R(e_1)$, so any other generator will have the form $g(y) \in k[y]$. But since every relation on x^3 and x^4 is generated by relations involving only x's, the same argument as in the previous example shows that e_1 is not killed by any element of $k[y]$. Thus $\mathrm{Ann}_R(e_1) = (x^3, x^4)$, and $Re_1 \cong R/(x^3, x^4)$.

Putting this together with our short exact sequence for M_λ, we can see that $\mathrm{Ass}_R M_\lambda = \{(x^3, x^4)\}$ which is independent of our choice of λ. This means that for this extension to be calm the modules $S \otimes M_\lambda$ can have only finitely many associated primes as they will all have to live in $\mathfrak{a}((x^3, x^4))$.

Tensoring our exact sequence with S we get

$$\cdots \to S \otimes Re_1 \to S \otimes M_\lambda \to S \otimes Re_2/(x^3e_2, x^4e_2) \to 0, \text{ which means}$$

$$\mathrm{Ass}_S \frac{Se_1}{\ker(Se_1 \to S \otimes M_\lambda)} \subseteq \mathrm{Ass}_S S \otimes M_\lambda.$$

The kernel of the map from Se_1 to $S \otimes M_\lambda$ consists of all relations on e_1 which exist in $S \otimes M_\lambda$, i.e. all relations on e_1 in the span over S (but not R) of x^3e_1, x^4e_1, x^3e_2 and $fe_1 - x^4e_2$. Since x^3 and x^4 already kill e_1 over R we discard anything in $(x^3, x^4)S$. We do have $x^2((y-\lambda)e_1 - x^4e_2) + x^3(x^3e_2) = x^2(y-\lambda)e_1$ in the S-span of our relations, so $x^2(y-\lambda)e_1$ is in the kernel. It is then clear that

$$\ker(Se_1 \to S \otimes M_\lambda) = x^2(y-\lambda)e_1S \subset Se_1.$$

Since $Se_1 \cong S/(x^3, x^4)$, this gives

$$\mathrm{Ass}_S S/(x^3, x^4, x^2(y-\lambda)) \subseteq \mathrm{Ass}_S S \otimes M_\lambda.$$

But

$$(x^3, x^4, x^2(y-\lambda)) = (x^2, x^3) \cap (x^3, x^4, y-\lambda), \text{ so}$$

$$\mathrm{Ass}(S/(x^3, x^4, x(y-\lambda))) = \{(x^2, x^3), (x^2, x^3, y-\lambda)\}.$$

Since the various $(x^2, x^3, y - \lambda)$'s are distinct for distinct values of λ, if k is infinite this gives a contradiction to the necessary finiteness of the attached sets of primes. (Again these primes collected by varying λ are precisely the primes of S which lie over the non-flat locus of S in R.) $\square$

CHAPTER VI

Open Questions

In this chapter we list some open questions remaining for future study.

Question VI.1. *Assume that S is flat over R with regular fibers, S is excellent, and R has an isolated singularity. Is $Ass_S H_I^i(S)$ finite for all ideals I of S?*

This is a generalization of Corollary IV.6 in two ways. First we have dropped the dimension restriction on the base ring, and second we have replaced the polynomial or power series extension with one which is merely flat with regular fibers.

Question VI.2. *Is $Ass_{(X_1,Y)} H_I^2(M)$ always finite for any finitely generated module, M, of pure dimension 4 over a ring of the form*

$$\frac{V[[X_1, X_2, X_3, Y]]}{(f)} \text{ or } \frac{k[[X_1, X_2, X_3, X_4, Y]]}{(f)}$$

where f is monic in Y and has a constant term divisible by X_1?

By Proposition III.7, this would show that $\operatorname{Ass}_R H_I^i(M)$ is always finite for local rings of dimension 4.

Question VI.3. *Is $Ass_R H_I^i(M)$ always finite where R is a non-local ring of dimension 3 and $ht(I) \leq 1$?*

Question VI.4. *Can we force* $Ass_R H^i_I(M)$ *to be finite if the height of* I *is large compared to the dimension of* R*? Specifically can we do this if our ideal,* I*, has* $ht(I) \geq dim(R) - 2$*?*

The second question would generalize Marley's result for ideals of height at least 2 in a local ring of dimension 4: see [Mar01, Proposition 2.8].

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ABSTRACT

Finiteness of Associated Primes of Local Cohomology Modules

by

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In this thesis we investigate when the set of primes of a local cohomology module is finite. We show that there are only finitely many primes associated to the local cohomology of any finitely generated module over a three-dimensional ring whose prime cyclic modules have S_2-fications with respect to an ideal whose height is at least two on that module. We show that a polynomial ring, R, over either an unramified regular local ring of mixed characteristic, or a two or three dimensional ring with a resolution of singularities Y_0, formed by blowing up an ideal of depth at least two where the sheaf cohomology of $\mathcal{O}_{Y_0}$ has finite length over the base ring, has $\mathrm{Ass}_R H^i_I(R)$ finite for any ideal $I \subset R$.

We also define a new class of extensions $R \to S$, called *calm*, where the associated primes of $S \otimes_R M$ over S are controlled by the associated primes of M over R for any R-module M. We show that calm extensions have many good properties including that compositions of calm maps are calm, calmness can be checked locally on open covers of $\mathrm{Spec}(R)$, and calmness persists after localization. We give various classes of rings whose extensions are all calm as well as some examples of extensions which are not calm.