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Introduction

The mathematical contributions of Craig Huneke ☆

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This issue of the *Journal of Algebra* grew out of a conference honoring Craig Huneke on the occasion of his sixty-fifth birthday, and the special editorial board for this issue coincides with the organizing committee for that conference. The contributors to this volume have a large overlap with the speakers at the conference, but the correspondence is far from exact.

During the conference, the authors of this introduction gave a joint talk highlighting some of the work of Craig Huneke. We have included here a version of what was said at the conference, but have supplemented it with additional remarks on Huneke's work and his enormous influence on the field of commutative algebra. We want to mention right away that Huneke has published over one hundred sixty papers with more than sixty co-authors. He has written two books and he has been an editor for four volumes. He has had twenty-five graduate students with another in progress. This does not take account of a huge number of mentees and colleagues who have benefited from his enormous generosity in sharing insight and ideas.

Huneke received his Bachelor of Arts degree from Oberlin College in 1973 and his Ph.D. in mathematics from Yale University in 1978. While his official advisor was Nathan Jacobson, David Eisenbud played a large role in mentoring his dissertation research. His first academic position was as a Junior Fellow in the Society of Fellows at the University of Michigan, where his significant interactions with Mel Hochster began, although they had met earlier. This position combined a postdoctoral research position and an assistant professorship. During this interval he also held a Visiting Scholar position at the Massachusetts Institute of Technology and was a Research Visitor at the Sonderforschungsbereich, Universität Bonn, both in 1980.

From 1981 through 1999, Huneke was on the faculty at Purdue University, first as Assistant Professor, from 1981 to 1984, then as Associate Professor, from 1984 to 1987, and as Professor, from 1987 to 1999. However, he spent 1981–82 as a National Science Foundation Fellow at the University of Illinois, he was a Research Visitor at the University of Virginia in 1984, he spent the academic year 1994–95 as a Visiting Professor at the University of Michigan, and, in 1998, he was a Fulbright Scholar and Visiting Professor at the Max Planck Institute, Bonn.

His collaboration with Hochster on the development of tight closure theory began during a conference at the University of Illinois in October 1986, and continued for two decades (cf. [1], [15–28]), including a one month period when both were visiting the University of Stockholm in the spring of 1988. During 1994–95, while visiting Michigan, he wrote the notes [48] for his ten CBMS Conference talks on tight closure: the conference took place in July 1995.

For the interval 1999–2012 he was the Henry J. Bischoff Professor at the University of Kansas. He was a Member at the Mathematical Sciences Research Institute, Berkeley, in the fall of 2002 and a Simons Research Professor at the same institution in the fall of 2012. Since 2012, he has held the position of Marvin Rosenblum Professor of Mathematics at the University of Virginia, where he also chaired the Mathematics Department for several years.

In the remainder of this introduction, we discuss several aspects of Craig Huneke's research contributions. These include some of our favorites among his theorems, as well as some related results in the literature. All of the work that we cite has been the starting point for a great deal of research by other experts in the field. We only discuss a fraction of his work, but it should be clear that his influence on the field of commutative algebra has been transformative. He has repeatedly opened up vast landscapes for exploration by many others.

In particular, we discuss some of Huneke's contributions in these overlapping areas:

- d-sequences and ideal powers
- Linkage
- Residual intersections
- Rees rings, reductions, and the core
- Tight closure
- Local cohomology
- Uniform behavior in Noetherian rings
- The Cohen-Macaulay property of R^+ in characteristic $p > 0$
- Symbolic powers

Since Hilbert functions and multiplicities were introduced, the behavior of powers of ideals has been an important topic in commutative algebra. This has led to the study of Rees rings, associated graded rings, integral dependence, and the relationship between powers and symbolic powers. Residual intersections arise in this context as well as in intersection theory and enumerative geometry; they generalize linkage or liaison, a classical method for classifying ideals and projective varieties and a rich source of examples.

Tight closure, related characteristic p ideas, and their relation to the absolute integral closure R^+ have become a significant part of the landscape in commutative algebra. They have also provided a foundation for recent breakthroughs in mixed characteristic. Huneke's research on the behavior of local cohomology and the questions he has raised have had a great influence on the field, and his work on uniform behavior in Noetherian rings has been an important tool for others as well as inspiration for further research. This area also provides an instance where tight closure methods may be applied, as does a significant part of his work on the behavior of symbolic powers.

1. d-sequences and ideal powers

Huneke introduced two important notions generalizing regular sequences, d-sequences and weak d-sequences, and developed their properties as a means to study powers of ideals, see [31–34,39,44]. In many cases, he was able to determine depths of ideal powers and to prove the equality of powers and symmetric or symbolic powers. Let $a_1, \ldots, a_n$ be elements of a Noetherian ring that generate an ideal I minimally. According to one of the equivalent definitions, these elements form a *d-sequence* if $\big((a_1, \ldots, a_{i-1}) : a_i\big) \cap I =$

$(a_1, \ldots, a_{i-1})$ for $1 \le i \le n$ ([39]). Huneke, and independently Valla, proved that in this case the symmetric powers and the powers of I coincide, in other words, the symmetric algebra $\mathrm{Sym}(I)$ and the Rees algebra $\mathcal{R}(I)$ are naturally isomorphic, see [31,81]. In addition, he showed that the associated graded ring $\mathrm{gr}_I(R)$ is Cohen-Macaulay if the sequence is a *Cohen-Macaulay d-sequence* in a Cohen-Macaulay ring ([32]). Whenever $\mathrm{gr}_I(R)$ is Cohen-Macaulay, then the equality of the powers and symbolic powers as well as the asymptotic depth of the powers can be read easily from the (local) analytic spreads of I, see [36] and the joint work with Eisenbud [9].

As it turned out later, there is an interesting connection between d-sequences and the approximation complex $\mathcal{M}(I)$ of Simis and Vasconcelos, which can also be used to study symmetric algebras and Rees algebra, see [76,77,13,14]. In fact, the $\mathcal{M}$-complex relates to d-sequences much like the Koszul complex relates to regular sequences: The $\mathcal{M}$-complex of an ideal I in a Noetherian local ring with infinite residue field is acyclic if and only if I can be generated by a d-sequence, see Huneke's [44] for an earlier result and [14] for the general case. The acyclicity of the $\mathcal{M}$-complex is implied, mainly, by depth conditions on the Koszul homology modules of the ideal and, in particular, by the Cohen-Macaulayness of these modules, in which case I is said to be *strongly Cohen-Macaulay*, see [76,77]. This motivates the search for classes of strongly Cohen-Macaulay ideals.

2. Linkage

Linkage, or liaison, has been used since the nineteenth century as a method for classifying projective varieties, in particular curves in $\mathbb{P}^3$, by authors such as Max Noether, Severi, Dubreil, Gaeta, and many others. The subject was reintroduced in the language of modern algebra by Peskine and Szpiro ([74]). Two proper ideals I and K of a Noetherian ring are said to be *linked*, $I \sim K$, if there exists a regular sequence $\underline{\alpha}$ so that $K = (\underline{\alpha}) : I$ and $I = (\underline{\alpha}) : K$. The ideals I and K are in the same (*even*) *linkage class* if $I = I_0 \sim I_1 \sim \cdots \sim I_n = K$ for some (even) integer n, and I is *licci* if it belongs to the *li*nkage *c*lass of a *c*omplete *i*ntersection ideal. Standard examples of licci ideals include perfect ideals of grade 2 and perfect Gorenstein ideals of grade 3. A goal of the subject is to classify linkage classes or, at least, to establish properties of licci ideals. With the next result, Huneke identified large classes of strongly Cohen-Macaulay ideals and provided one of the first tools to show that a perfect ideal is not licci.

Theorem 2.1. *[38] Let I and K be ideals of a Cohen-Macaulay ring that are in the same even linkage class. Then I is strongly Cohen-Macaulay if and only if K is strongly Cohen-Macaulay. In particular, every licci ideal is strongly Cohen-Macaulay.*

Strengthening Theorem 2.1, Huneke introduced numerical invariants measuring the deviation of an ideal from being strongly Cohen-Macaulay, and he proved that these invariants too are constant across the even linkage class, see [42,43].

Huneke's work on linkage also includes [35,40,2,55] and the articles [60–67] with Ulrich, where the notions of generic and universal linkage are used in a systematic study of linkage classes and licci ideals. It is proved, for instance, that if a factor ring of a regular local ring by a licci ideal is not a complete intersection then its non-complete-intersection locus has codimension at most 7, and that if the factor ring is not Gorenstein then its non-Gorenstein locus has codimension at most 4. Other results relate the shifts in homogeneous free resolutions to properties of linkage classes:

Theorem 2.2. *[61] Let $I' \subset R' = k[x_1, \ldots, x_d]$ be a nonzero homogeneous perfect ideal with minimal homogeneous resolution*

$$0 \to \oplus_i R'(-n_{gi}) \longrightarrow \quad \ldots \quad \longrightarrow \oplus_i R'(-n_{1i}) \longrightarrow I' \to 0\,.$$

Write $R = R'_{(x_1, \ldots, x_d)}$ and $I = I'R$. If

$$\max\{n_{gi}\} \leq (g-1)\min\{n_{1i}\},$$

then for every ideal K in the even linkage class of I,

$$\mu(K) \geq \mu(I) \quad \textit{and} \quad r(R/K) \geq r(R/I),$$

where μ and r denote minimal number of generators and type, respectively. In particular, I cannot be licci.

Whereas there is only one linkage class of arithmetically Cohen-Macaulay curves in $\mathbb{P}^3_k$ (or arithmetically Gorenstein curves in $\mathbb{P}^4_k$, respectively), Theorem 2.2 can be used to show, for instance, that there are infinitely many smooth arithmetically Cohen-Macaulay curves in $\mathbb{P}^4_k$ (or arithmetically Gorenstein curves in $\mathbb{P}^5_k$, respectively) that belong to different linkage classes.

3. Residual intersections

Residual intersections, a vast generalization of linkage, appear in intersection theory, enumerative geometry, and the study of Rees rings, for instance. A proper ideal K of a Noetherian ring is said to be an *s-residual intersection* of an ideal I if $K = (\alpha_1, \ldots, \alpha_s) : I$, where $\alpha_i \in I$, and $\operatorname{ht} K \geq s$. Huneke was the first to prove results about the Cohen-Macaulayness of residual intersections and to recognize a connection with the depth of Koszul homology modules:

Theorem 3.1. *[41] Let I be an ideal of a Cohen-Macaulay ring R and assume that I is strongly Cohen-Macaulay and satisfies $\mu(I_p) \leq \operatorname{ht} p$ for every $p \in V(I)$ with $\operatorname{ht} p \leq s-1$. If K is any s-residual intersection of I, then R/K is Cohen-Macaulay and $\operatorname{ht} K = s$.*

Moreover, in [64] the canonical module of R/K is computed and this information is used to remove, in some cases, the above assumption on the local numbers of generators of I. Meanwhile, there have been many other contributions, by a number of authors, to this subject that has its roots in Huneke's [41].

In [41] Huneke also proved that if an ideal I of a Cohen-Macaulay ring satisfies $\mu(I_p) \leq \operatorname{ht} p$ for *every* $p \in V(I)$, then the defining ideal of the extended symmetric algebra of I is a residual intersection. Hence, if in addition I is strongly Cohen-Macaulay, then this algebra is Cohen-Macaulay by Theorem 3.1, which implies once more the facts, mentioned in Section 1, that the rings $\operatorname{Sym}(I)$ and $\mathcal{R}(I)$ are isomorphic and that $\operatorname{gr}_I(R)$ is Cohen-Macaulay. Whereas the isomorphism $\operatorname{Sym}(I) \cong \mathcal{R}(I)$ requires that $\mu(I_p) \leq \operatorname{ht} p$ for every $p \in V(I)$, it is interesting to ask when $\operatorname{gr}_I(R)$ or $\mathcal{R}(I)$ are Cohen-Macaulay without this assumption. Huneke, in collaboration with Huckaba [29,30], was the first to address this question for ideals that are not necessarily integral over complete intersections.

4. Rees rings, reductions, and the core

To deal with ideals whose number of generators exceeds the dimension of the ambient ring, Huckaba and Huneke pass to a minimal reduction of the ideal. Any ideal I of a Noetherian local ring R with infinite residue field has a minimal reduction, a minimal ideal over which I is integral. Every such minimal reduction has the same number of generators, called the *analytic spread* of I and denoted by $\ell(I)$, and indeed $\ell(I) \leq \dim R$; for these and other facts about integral dependence of ideals, see the excellent book [80] of Huneke and Swanson. The connection between I and a minimal reduction J is measured by the *reduction number*, and when this number is small one can hope for a transfer of properties from J back to I.

The main results of [29,30] treat the case $\ell(I) \leq g + 2$, where $g = \operatorname{ht} I$. Huckaba and Huneke prove that I has reduction number ≤ 1, i.e., $I^2 = JI$ for some minimal reduction J, if, for instance, the following hypotheses are satisfied:

- R is Cohen-Macaulay, I is unmixed, $g \geq 1$, I is a complete intersection locally in codimension $g + 1$, and $\ell(I) \leq g + 1$.

With the same hypotheses they prove that if R is regular then the powers and the symbolic powers of I coincide and if R/I is almost Cohen-Macaulay then $\mathcal{R}(I)$ and $\operatorname{gr}_I(R)$ are Cohen-Macaulay. Analogous statements are proved if the above hypotheses are replaced by the assumptions:

- R is Gorenstein, R/I is Cohen-Macaulay, $g \geq 2$, I is a complete intersection locally in codimension $g + 2$, and $\ell(I) \leq g + 2$.

These results are difficult, and they have triggered a great deal of work throughout the 1990s addressing the Cohen-Macaulayness of Rees algebras for ideals that have arbitrary analytic spread.

Unlike the integral closure, the largest ideal integral over an ideal, minimal reductions are not unique. This leads to the definition of the *core* of an ideal, which is the intersection of all its minimal reductions ([75]). The core is related to adjoints or multiplier ideals and to the Briançon-Skoda theorem, which, in its simplest form, says that $I^\ell \subset \operatorname{core}(I)$ for any ideal I of analytic spread ℓ in a regular local ring. The core of an ideal is notoriously difficult to determine. The first substantial result in this direction is due to Huneke and Swanson:

Theorem 4.1. *[58] Let $I \neq 0$ be an integrally closed ideal in a two-dimensional regular local ring with infinite residue field. Then*

$$\operatorname{core}(I) = I \cdot \operatorname{Fitt}_2(I) = J^2 : I\,,$$

where J is any minimal reduction of I.

This theorem also established, for the first time, a connection with Lipman's adjoints, namely $\operatorname{core}(I) = I \cdot \operatorname{adj}(I) = \operatorname{adj}(I^2)$ if $\operatorname{ht} I = 2$. After [58], there has been a growing literature about cores, including an article by Huneke and Trung ([59]).

5. Tight closure

Tight closure provides a systematic method of proving results in prime characteristic $p > 0$, and in equal characteristic zero by reduction to characteristic p. Since its introduction in the late 1980s, it has had a dramatic effect on the field of commutative algebra. A key point is that over a regular ring, every ideal is tightly closed and every submodule of every finitely generated module is tightly closed, and that this closure is always contained in the integral closure (see Section 4), but is typically substantially smaller. Tight closure gives unified proofs and strong generalizations of many major theorems in commutative algebra. One is the fact that rings of invariants of linearly reductive groups acting on regular rings are Cohen-Macaulay, and, more generally, if R is a direct summand as an R-module of a regular R-algebra S, then R is Cohen-Macaulay. This contains the result that normal subrings of polynomial rings generated by monomials are Cohen-Macaulay. Another is the Briançon-Skoda theorem, which was first proved in equal characteristic zero by Briançon and Skoda and in complete generality by Lipman and Sathaye [68]. The original theorem asserts that if an ideal I of a regular ring R is generated by at most n elements (up to integral closure), then $\overline{I^n}$, the integral closure of I^n, is contained in I. The tight closure version in equal characteristic does not require that the ring R be regular. It asserts that $\overline{I^n}$ is contained in I^*, the tight closure of I. See also Theorems 7.2 and 9.1 for related results and alternative formulations.

Tight closure can also be used to give new proofs of the direct summand conjecture, existence of big Cohen-Macaulay modules, and to generalize the Ein-Lazarsfeld-Smith theorem on symbolic powers [7] to positive prime characteristic, see Section 9. Various characteristic p notions related to tight closure have led to many connections between the study of singularities in equal characteristic zero and in positive prime characteristic p.

6. Local cohomology

Huneke has played a fundamental role in shaping the study of local cohomology. His paper [46] has been the inspiration for many other researchers. He did fundamental work on vanishing theorems in [53], and his results on local cohomology of regular rings of positive prime characteristic, joint with Sharp [57], were the pre-cursor to a huge explosion of further research by Lyubeznik (see, for example [69–71]) and many, many others. Earlier, related work was done by Hartshorne and Speiser [12]. The following result is from [57]:

Theorem 6.1. *Every local cohomology module $H^i_I(R)$ of a regular local ring R of prime characteristic $p > 0$ has finitely many associated primes, finite Bass numbers, and is injective if supported only at the maximal ideal.*

Note that it is an open question whether every local cohomology module $H^i_I(M)$ of a Noetherian module over a Noetherian ring R has closed support, i.e., has only finitely many *minimal* primes. This question was raised by Huneke in [46]. We conclude the section with two results from Huneke's joint work with Katz and Marley [50] on this topic.

Theorem 6.2. *The support of $H^2_I(M)$ is closed when I has cohomological dimension at most two or R is local of dimension at most four.*

Theorem 6.3. *Let R be a Noetherian ring containing a field of characteristic zero. Let $I = (x_1, \ldots, x_n) \subset R$ where $n \geq 6$. Then there exists a 2×3 matrix A with entries from R such that $H^n_I(R) \cong H^3_{I_2(A)}(R)$.*

7. Uniform behavior in Noetherian rings

Following earlier work of Zariski [83], Eisenbud-Hochster [8], Duncan-O'Carroll [5,6], and O'Carroll [73] (all for the case where I is maximal or principal), Huneke obtained the following beautiful results [47].

Theorem 7.1 *(Uniform Artin-Rees. Huneke). Let R be a Noetherian ring and let $N \subset M$ be two finitely generated R-modules. If R satisfies at least one of the conditions below, then there exists an integer k such that for all ideals I of R, and all $n \geq k$, $I^n M \cap N \subset I^{n-k} N$.*

(1) *R is essentially of finite type over a Noetherian local ring.*

(2) *R is a ring of prime characteristic $p > 0$ and F-finite.*
(3) *R is essentially of finite type over $\mathbb{Z}$.*

Theorem 7.2 *(Uniform Briançon-Skoda. Huneke). Let R be a Noetherian reduced ring. If R satisfies at least one of the conditions below, then there exists an integer k such that for all ideals I of R, and all $n \geq k$, $\overline{I^n} \subset I^{n-k}$.*

(1) *R is essentially of finite type over an excellent Noetherian local ring.*
(2) *R is of prime characteristic $p > 0$ and F-finite.*
(3) *R is essentially of finite type over $\mathbb{Z}$.*

8. The Cohen-Macaulay property of R^+ in characteristic $p > 0$

If R is a domain, R^+ denotes the integral closure of R in an algebraic closure of its fraction field, called the *absolute integral closure* of R. It is unique up to non-unique isomorphism. It is a largest domain extension that is integral over R.

Let R be a Noetherian local domain of characteristic $p > 0$. The first theorem, proved by Hochster and Huneke [20], states that if R is excellent, then R^+ is a big Cohen-Macaulay algebra, in the sense that every system of parameters in R is a regular sequence in R^+.

The result of Huneke and Lyubeznik [54] is this:

Theorem 8.1 *(Huneke-Lyubeznik). Let (R, m), of Krull dimension d, be the homomorphic image of a Gorenstein local ring. Then there is a module-finite extension domain S of R such that all the maps $H^i_m(R) \to H^i_m(S)$ are zero for $i < d$.*

This gives a new, simpler proof of the theorem on R^+ in the main cases (the hypotheses are different, but either result may be used in the most frequent cases).

9. Symbolic powers

The nth-*symbolic power* of an ideal I in a Noetherian ring R is defined as $I^{(n)} := R \cap I^n_W$, where W is the complement of the union of the associated primes of I. If I is a prime ideal in a polynomial ring over a field for instance, then $I^{(n)}$ consists of the polynomials that vanish with order at least n at every closed point of $V(I)$, according to the Zariski-Nagata theorem. Although symbolic powers are ubiquitous in algebra and geometry, they are not well understood and, in particular, their relation to ordinary powers, which they always contain, is mysterious. This problem is a recurring theme in Huneke's work.

As mentioned in Sections 1 and 4, Huneke made great strides in identifying cases where powers and symbolic powers coincide. He also observed, in general, that if $I^{(n)} = I^n$ for all (or infinitely many) n then $\ell(I_p) < \operatorname{ht} p$ for every prime $p \in V(I)$ not contained in an associated prime of I, and that the converse holds whenever $\mathrm{gr}_I(R)$ is Cohen-Macaulay, see [36,9]. If R is an excellent domain, the condition on the local analytic spreads alone implies that the symbolic Rees ring $\bigoplus_{n\geq 0} I^{(n)}$ is Noetherian ([36]). Whether symbolic Rees rings are Noetherian or not has been investigated for many reasons; for instance, a one-dimensional prime ideal in a Noetherian local ring is a set-theoretic complete intersection if its symbolic Rees ring is Noetherian ([4]). In [37,45,56] Huneke gave effective criteria for the Noetherianness of symbolic Rees rings and applied them masterfully.

The best affirmative results one could expect in general are comparison theorems between powers and symbolic powers. In this direction Swanson showed that for any given ideal I in an excellent normal ring, there exists an integer h such that $I^{(hn)} \subset I^n$ for all n, see [79] and, for this particular type of statement, [52]. Ein, Lazarsfeld, and Smith proved the surprising result that in a regular ring R of finite type over a field of characteristic zero, one can take h to be the biggest height of an associated prime of the (radical) ideal I ([7]). In particular, one can take h to be $\dim R$, which does not depend on I! Hochster and Huneke proved the corresponding result in characteristic $p > 0$, using tight closure techniques ([26]). In fact, their work applies to rings containing a field of arbitrary characteristic and to rings that are not necessarily regular:

Theorem 9.1. *[26] Let R be a Noetherian ring containing a field k, let I be an ideal, and let h be the biggest height of an associated prime of I.*

(1) *If R is regular, then $I^{(hn)} \subset I^n$ for all n.*
(2) *If R is a domain and a finitely generated geometrically reduced k-algebra, with Jacobian ideal $\mathcal{J}$, then $\mathcal{J}^{n+1} I^{(hn)} \subset I^n$ for all n.*

The proof is by reduction to prime characteristic. In characteristic p, for (1) it is shown that $I^{(hn)} \subset (I^n)^* = I^n$. In (2) it is also used that $\mathcal{J}$ can be generated by completely stable test elements. Recently, using perfectoid algebras, Ma and Schwede obtained an analogue of Theorem 9.1(1) in mixed characteristic ([72]).

An immediate consequence of Theorem 9.1(1) is a result of Skoda [78] and Waldschmidt [82] on initial degrees of symbolic powers, originally proved with analytic methods for $k = \mathbb{C}$; it says that if I is the defining ideal of a finite set of points in $\mathbb{P}^N_k$, then $\mathrm{indeg}(I^{(n)}) \geq \frac{\mathrm{indeg}(I)}{N} \cdot n$ for all n, where $\mathrm{indeg}(H)$ denotes the smallest degree of a nonzero form in a homogeneous ideal H. The containment $I^{(hn)} \subset I^n$ in Theorem 9.1(1) is not always optimal. This led to a number of hypothetical improvements proposed by Huneke, Harbourne, and others. One of these improvements implies a sharper version of the inequality of Skoda and Waldschmidt that was proved by Chudnovsky [3] for $N = 2$ and conjectured in general. Another question of Huneke is whether $I^{(3)} \subset I^2$ for any prime ideal I of height 2 in a regular local ring. These problems have attracted much attention. Huneke, with his coauthors, solved them in many important cases, see [27,11,10].

If the ambient R is not necessarily regular, one can still hope for a uniform comparison between the powers and the symbolic powers of all prime ideals of R, at least when R is an excellent normal local ring. Katz, Huneke, and Validashti prove this for isolated singularities in equal characteristic:

Theorem 9.2. *[51] Let R be an analytically irreducible Noetherian local ring with an isolated singularity and assume that R is essentially of finite type over a field of characteristic zero or that R is of prime characteristic and F-finite. Then there exists an integer h such that for every prime ideal I of R, and every n, $I^{(hn)} \subset I^n$.*

This result is hard! In addition to [26], it uses both existing results on uniform behavior, Theorems 7.1 and 7.2, and other ingenious techniques. In [52,49] Huneke and his coauthors study descent and ascent under integral extensions and they prove in particular that the conclusion of Theorem 9.2 also holds if, essentially, R is a finite Abelian extension of a regular domain containing a field. The problem appears to be very difficult, and many open questions remain.

We hope that this account, though being far from complete, illustrates the tremendous breadth and depth of Huneke's work. Huneke advanced commutative algebra through his seminal ideas and insights, and he initiated important new developments in the field. He inspired generations of commutative algebraists and provided ample research opportunities for many. Commutative algebra would not be the same without him. We owe an inestimable debt to Craig Huneke!

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Melvin Hochster
Department of Mathematics,
University of Michigan, Ann Arbor,
MI 48109–1043, USA
E-mail address: hochster@umich.edu

Bernd Ulrich *
Department of Mathematics,
Purdue University, W. Lafayette,
IN 47907–2067, USA
E-mail address: bulrich@purdue.edu

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* Corresponding author.