TICHT CLOSURE IN EQUAL CHARACTERISTIC 0 AND OPEN QUESTIONS IN ALL CHARACTERISTICS

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July 25, 2022

My thanks to the organizers, both for the enormous amount of work they did, and for inviting me to speak.

We discuss how to extend tight closure theory in prime characteristic $p > 0$ to Noetherian rings that contain the rational numbers. We then consider several important open questions that remain.

In characteristic p, some results (particularly those depending on the existence of test elements) are limited to the case of F-finite rings or to the case of algebras essentially of finite type over an excellent semi-local ring. For simplicity, our discussion of tight closure in characteristic p tacitly assumes that we are in one of these cases. In our discussion of equal characteristic 0, we assume instead that all rings are locally excellent, simply because that is what we need.

For those unfamiliar with excellent rings, we mention that these include all algebras that arise as localizations of finitely generated algebras over a complete local ring, as well as F-finite rings and rings of convergent power series. This class is closed under taking quotients, localizations, and forming finitely generated algebras. By a local ring we always mean a *Noetherian* ring with a unique maximal ideal. In the local case, we sometimes assume that given rings in theorems are *equidimensional*, i.e., the quotient of the ring by any minimal prime has the same dimension as the ring.

"Philosophical" remarks about tight closure

1. Tight closure "remedies" the fact that not every local ring is Cohen-Macaulay. It gives partial control of what happens when a system of parameters is not a regular sequence.

2. It is also a partial remedy for the fact that not every ring is regular. Some properties of regular rings that fail in general hold "up to tight closure" in rings that are not regular, like the Briançon-Skoda theorem. Rings in which every ideal is tightly closed have milder singularities than other rings.

3. One of the most important uses of tight closure occurs in rings that are regular, or weakly F-regular (every ideal is tightly closed). In these rings, tight closure gives an "easier" way of testing ideal membership, enabling one to prove some deep theorems (e.g., about symolic powers).

We are now ready to state some properties of tight closure. As mentioned earlier, in char. p some of them need that the ring is F-finite or essentially of finite type over a complete semi-local ring (char. p) or locally excellent (char. 0). These hypotheses are often unnecessary but avoid many technicalities. We call a Noetherian ring R *admissible* if it is equal characteristic and has the appropriate properties specified above. This is *not* standard terminology in the literature. In [HH] a notion of tight closure is developed for the class of Noetherian K-algebras, where K is a given field of equal characteristic zero. This kind of tight closure is denoted $*^K$ in [HH]. We shall focus here on the notion that corresponds to the case $K = \mathbb{Q}$, which is also called *equational tight closure* in [HH]. Here, we shall simply denote it $*$.

Tight closure is an operation on submodules of finitely generated modules defined over any admissible base ring (and more generally). We shall denote the tight closure of N in M by N^* or N_M^*. If $N = I$ is an ideal of R then M is understood to be R unless otherwise specified: this is a very important case. If $N \subseteq M$ are R-modules and $R \to S$ is a homomorphism we shall write M_S for $S \otimes_R M$ and $\langle N_S \rangle$ for the image of N_S in M_S. This is an abuse of notation since N_S depends on $N \subseteq M$ and not just on N. In the case when M is free, forming $\langle N_S \rangle$ is easily seen to be parallel to extending an ideal of R to an ideal of S (which is the case $M = R$).

Properties of tight closure II

By a *complete local domain* of R we mean a ring obtained by localizing R at a prime ideal, completing, and then killing a minimal prime. We define the *minheight* of $I \subseteq R$ as the minimum of the heights of $I(R/P)$ for minimal primes P of R. We write $\overline{J}$ for the integral closure of the ideal J.

We want to emphasize that the following result holds in characteristic p and characteristic 0.

Theorem (Properties of tight closure)

Let R, S be admissible reduced rings, $R \to S$ a homomorphism, and let $N \subseteq M \subseteq Q$ be finitely generated R-modules. Tight closures are taken in M unless otherwise specified. Let u denote an element of M, and I an ideal of R.

1. $u \in N_M^*$ if and only if $u + N \in 0_{M/N}^*$.
2. $N_M^* \subseteq N_Q^*$, $N_Q^* \subseteq M_Q^*$, and $N_M^{**} = N_M^*$.
3. (Persistence of $*$.) If $u \in N_M^*$ then $1 \otimes u \in \langle N_S \rangle_{M_S}^*$.
4. $u \in N_M^*$ iff for every complete local domain B of R, $1 \otimes u \in \langle N_B \rangle_{M_B}^*$.
5. If R is a regular ring, $N_M^* = N$ for all $N \subseteq M$.
6. (Generalized Briançon-Skoda theorem) If I has at most n generators then for all $k \in \mathbb{N}$, $\overline{I^{n+k}} \subseteq (I^{k+l})^*$.

7. (Capturing colon ideals) If $I = (x_1, \ldots, x_n)R$ is such that $\text{minheight}(I) \geq n$ and $I_{n-1} = (x_1, \ldots, x_{n-1})R$, then $I_{n-1} :_R x_n \subseteq I_{n-1}^*$. (Better: $I_{n-1}^* :_R x_n = I_{n-1}^*$.)

8. (Phantom acyclicity criterion) Suppose that $G_\bullet$ is a finite free complex of length n over R with rank $G_i = b_i$, and let $r_i = \sum_{j \geq i}(-1)^{j-i}b_i$. If $G_\bullet$ is such that the rank of the i th map is r_i for $1 \leq i \leq n$ and such that the minheight of the ideal gen. by the r_i size minors of a matrix for $d_i : G_i \to G_{i-l}$ is $\geq i$, then for all $i \geq 1$, $\text{Ker}(d_i) \subseteq \left(\text{Im}(d_{i+1})\right)^*_{G_i}$ (and then $G_\bullet$ is called *phantom acyclic.*)

9. If S is a module-finite extension of R then $u \in N_M^*$ over R if and only if $u \in (N_S)^*_{M_S}$ Moreover, $IS \cap R \subseteq I^*$.

Weakly F-regular and F-rational rings I

If one uses the depth of R on I instead of the minheight, the criterion on part 8 is the Buchsbaum-Eisendbud acyclicity criterion. If R is local and equidimensional, we may use height instead of minheight in the statement.

We emphasize that all of these results are correct *in both char. p and in equal char. 0*. For the moment we continue to assume that * is understood in equal char. 0. Then we will explain how to define it.

From the properties of tight closure discussed above, one gets many striking results. Call a ring *weakly F-regular* if every ideal is tightly closed. Call a ring *F-rational* if every ideal I generated by parameters (i.e., elements that are part of a system of parameters in every local ring at a prime containing I) is tightly closed.

It is not difficult to show that if R is weakly F-regular then every submodule of every finitely generated module is tightly closed. R is called *F-regular* if all of its local rings are weakly F-regular. F-rational, weakly F-regular, and F-regular are known to be equivalent conditions if the ring is Gorenstein.

By the *regular closure* N_M^{reg} of N in M we mean the set of all $u \in M$ such that for every map of R to a regular ring S, $1 \otimes u \in \langle N_S \rangle_{M_S}$.

The following are easily deduced from properties we have already discussed.

Theorem

Let R be an admissible ring of equal characteristic. Let $N \subseteq M$ be finitely generated R-modules and let I be an ideal of R.

1. *$N^* \subseteq N^{\mathrm{reg}}$ (this is known to be strict), and $I^{\mathrm{reg}} \subseteq \bar{I}$. (Integral closure is tested by mapping to DVR's.)*

2. *Weakly F-regular rings are normal and Cohen-Macaulay. In fact, even F-rational rings are normal and Cohen-Macaulay.*

3. *A direct summand (or a pure subring) of a weakly F-regular ring is weakly F-regular. Hence, a direct summand (or pure subring) of a regular ring is Cohen-Macaulay.*

4. *A weakly F-regular ring is a direct summand of every module-finite extension. (Only of interest in char. $p > 0$.)*

The vanishing theorem for maps of Tor

Theorem (Vanishing theorem for maps of Tor)

Let A be a regular equicharacteristic domain, let R be module-finite extension domain of A, and let $R \to S$ be any map to a ring that is regular (or admissible and weakly F-regular). Then for every A-module M and every $i \geq 1$, the map $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^A(M, S)$ is zero.

Proof.

Reduce to the case where M is fin. gen. Take a finite A-free resolution $G_\bullet$ of M. Since $G_\bullet$ satisfies the Buchsbaum-Eisenbud acyclicity criterion, $R \otimes_A G_\bullet$ satisfies the phantom acyclicity criterion: hence, $R \otimes_A G_\bullet$ has phantom homology ($A \to R$ does not preserve depth but does preserve height). By the persistence of tight closure, the image z of a cycle in $R \otimes_A G_\bullet$ in $S \otimes_A G_\bullet$ is still in the tight closure of the boundaries. But since S is weakly F-regular, z is now a boundary. □

This is an extremely powerful vanishing theorem, and is now known even in mixed characteristic.

The case where R is local and S is the residue field of R is equivalent to the direct summand conjecture.

The case where R is a direct summand of S easily yields another proof that direct summands of regular rings are Cohen-Macaulay.

We next give the definition of tight closure for rings containing a field of characteristic 0.

Descent. Let R be a finitely generated algebra over $\mathbb{Q}$. A presentation of R can be given using only finitely many coefficients. Hence, all coefficients are in a ring $A = \mathbb{Z}_a \subseteq \mathbb{Q}$, and we can write $R = \mathbb{Q} \otimes_A R_A$, where R_A is a finitely generated A-algebra. In the discussion that follows, we may invert finitely many more integers. Thus, A may change, becoming larger. The ring R is the direct limit of all the rings R_A as A varies through all subrings of $\mathbb{Q}$ of the form $\mathbb{Z}_a$. If $N \subseteq M$ are fin. gen. R-modules, we can find an inclusion of finitely generated R_A-modules $N_A \subseteq M_A$ that becomes $N \subseteq M$ when we tensor with $\mathbb{Q}$.

If $u \in M$ we can choose A so that there is an element $u_A \in M_A$ that maps to u in M. If $M = R$, and $N = I$, we can take $M_A = R_A$, and make A large enough to contain generators of I, which will generate an ideal $I_A \subseteq R_A$. If B is any A-algebra, we indicate a base change from A to B by replacing the subscript A by the subscript B.

By the lemma on generic freeness, we may also assume that finitely many given fin. gen. R_A-modules are A-free. Thus, we may assume, for example that R_A, I_A, R_A/I_A, M_A, N_A, and M_A/N_A are A-free. Once M_A/N_A is A free, we may tensor over A with any A-algebra B and if we started with $N_A \subseteq M_A$, we have that $N_B \subseteq M_B$. *We typically assume M_A, N_A, M_A/N_A are A-free.*

Once M_A is A-free, we have $M_A \subseteq M_{\mathbb{Q}} = M$, and

$$M = \bigcup_{A \subseteq A' \subseteq \mathbb{Q}} M_{A'},$$

where A' has the form $\mathbb{Z}_{a'}$.

It is worth keeping in mind that in any finitely generated $\mathbb{Z}$-algebra, the quotient by any maximal ideal is a finite field, and so has prime char. $p > 0$.

Now we can give the definition. If $N \subseteq M$ are finitely generated modules over an affine $\mathbb{Q}$-algebra, the element $u \in M$ is in N_M^* if for some (equivalently, all) sufficiently large $A = \mathbb{Z}_a \subseteq \mathbb{Q}$, there exist u_A, $N_A \subseteq M_A$ as above with N_A, M_A, M_A/N_A all A-free such that for all but finitely many prime integers p not dividing a, if $\kappa_p = \mathbb{Z}_a/p\mathbb{Z}_a$, we have $u_{\kappa_p} \in (N_{\kappa_p})_{M_{\kappa_p}}^*$ *working over the ring R_{κ_p}*, where tight closure here is in the char. p sense.

Let $R := \mathbb{Q}[X, Y, Z]/(X^3 + Y^3 + Z^3) = \mathbb{Q}[x, y, z]$ and let $I = (x, y)$. Then $z^2 \in I^*$. Here we may take $A = \mathbb{Z}$, $R_A = \mathbb{Z}[X, Y, Z]/(X^3 + Y^3 + Z^3) = \mathbb{Z}[x, y, z]$, $I_A = (x, y)R_A$ and $u_A = z^2 \in R_A$. The situations when we pass to κ_p for any $p \neq 3$ are very similar. In every case, we have $z^2_{\kappa_p} \in \left((x, y)R_{\kappa_p}\right)^*$. Hence, $z^2 \in (x, y)^*$ in R.

Let S be a locally excellent Noetherian ring containing $\mathbb{Q}$. We let $N \subseteq M$ be finitely generated S-modules and $u \in M$. We say that $u \in N_M^*$ if there exists an affine $\mathbb{Q}$-algebra R with a map $R \to S$ and an inclusion of finitely generated R-modules $N_R \to M_R$, an element $u_R \in M_R$ such that $u_S \in (N_S)_{M_S}^*$ in the sense of affine $\mathbb{Q}$-algebras and such that after base change $R \to S$,

1 $M_S \cong M$ and $N_S \to M_S = M$ maps N_S onto N.

2 $1 \otimes u_R \in S \otimes M_R$ maps to $u \in M$.

That is, all instances of tight closure arise from instances over an affine $\mathbb{Q}$-algebra followed by a base change (and are forced if we want persistence to hold).

We next note:

Theorem

The notion of $*$ for arbitrary locally excellent $\mathbb{Q}$-algebras agrees with the notion we originally defined for affine $\mathbb{Q}$-algebras.

The next result is rather difficult. It depends on a result of Artin and Rotthaus that is a special case of Néron-Popescu desingularization. Recall that a *complete local domain* of R is the result of killing a minimal prime in the completion of the localization of R at a prime ideal.

Theorem

Let R be a locally excellent Noetherian ring containing $\mathbb{Q}$. Let $N \subseteq M$ be finitely generated R-modules, and $u \in M$. Then the following two conditions are equivalent:

1. The element $u \in N_M^*$.
2. For every complete local domain C of R, we have that $u_C \in (\langle N_C \rangle)_{M_C}^*$ working over C.

This means that our notion of tight closure in equal char. 0 is completely determined by its behavior for complete local domains.

Theorem

The following rings are F-regular both in equal char. p and in equal char. 0.

1. *Normal rings generated by monomials in the indeterminates over a polynomial ring.*

2. *Let X be an $r \times s$ matrix of indeterminates over a field K, with $1 \leq r \leq t \leq s$. Let $K[X]$ be the polynomial ring generated over K by the entries of X. Let $I_t(X)$ denote the ideal generated by the size t minors of X in $K[X]$.*

   (a) *$K[X]/I_t(X)$ (a generic determinantal ring).*

   (b) *The subring of $K[X]$ generated by the size r minors (the homogeneous coordinate ring of a Grassmann variety).*

Definition

Let R be a domain of char. $p > 0$. If R is F-finite, we define R to be *strongly F-regular* if for every $c \neq 0$ in R, there exists $e \geq 1$ such that the R-linear map $\alpha : R \to R^{1/p^e}$ with $1 \mapsto c^{1/p^e}$ splits over R. Let β be the splitting of α. When this happens for a choice $c \neq 0$ in R, one also gets that R is F-split and that the same condition holds for all $e' \geq e$.

Then:

Theorem

If R is strongly F-regular, then every submodule of every module is tightly closed (finite generation is not required).

Proof.

In the case of an ideal $I \subseteq R$, $cu^q \in I^{[q]}$ for all $q \gg 0$ implies $c^{1/q}u \in IR^{1/q}$. Applying the splitting β that exists for $q \gg 1$ yields $u \in I$. We may give exactly the same argument when R is replaced by an arbitrary direct sum of copies of R and I by an R-submodule of that free module. $\qquad\square$

Thus, strongly F-regular rings are weakly F-regular, and, hence, normal.

The intersection of the ideals $N :_R N_M^*$ over all pairs of finitely generated modules $N \subseteq M$ is called the *test ideal* of R. The elements in the test ideal that are not in any minimal prime are called *test elements*. A test element is called *completely stable* if its image in the completion of any localization of R at a prime ideal is still a test element.

Char. p: If R is reduced and is F-finite or essentially of finite type over an excellent semilocal ring, $c \in R$ is not in any minimal prime, and R_c is regular (strongly F-regular suffices), *then c has a power that is completely stable test element*.

If $R = K[x_1, \ldots, x_n]/I$ where K is an algebraically closed field, and R is reduced of pure dimension d, so that all minimal primes of I have the same height $h := n - d$, it is well known that the *Jacobian ideal* defines the singular locus in R. If I has generators $f_1, \ldots, f_m$ the Jacobian ideal $\mathcal{J}$ is generated in R by the images of the size h minors of the matrix $(\partial f_i/\partial x_j)$. By results in [HH], [Jia], in *all* characteristics:

Theorem

Let notation be as above. Then the elements of $\mathcal{J}$ are in the test ideal of R, and this remains true if we replace R by the completion of its localization at any prime ideal.

There are similar results in [Jia] when
$R = K[[x_1, \ldots, x_n]][y_1, \ldots, y_k]/I$.

Open question

Let B denote a finitely generated domain over $\mathbb{Z}$ with fraction field K, and let R_B denote a finitely generated B-algebra. Is it true that R_K is weakly F-regular if and only if for a Zariski dense open set of the maximal ideals μ of B, R_{κ_μ} is weakly F-regular, where $\kappa_\mu = B/\mu$?

A Noetherian domain R is called a *splinter* if it is a direct summand as an R-module of every module-finite extension ring. $\mathbb{Q}$-algebras are splinters iff they are normal. In char. $p > 0$, being a splinter is much more restrictive. It is known that:

$$\textit{weakly F-regular} \Rightarrow \textit{splinter} \Rightarrow \textit{F-rational}$$
$$\Rightarrow \textit{(Cohen-Macaulay and normal).}$$

Open question

Is a locally excellent Noetherian domain of prime characteristic $p > 0$ a splinter if and only if it is weakly F-regular?

The issue is local. This is true if R is Gorenstein or $\mathbb{Q}$-Gorenstein.

Open question

In prime characteristic $p > 0$, is a weakly F-regular F-finite ring necessarily strongly F-regular?

This is known in if the ring R is an $\mathbb{N}$-graded finitely generated K-algebra with $R_0 = K$ or if the ring is Gorenstein. It would suffice to prove the result for complete F-finite local rings.

Open question

Does every reduced excellent ring of char. $p > 0$ have a test element? More specifically, is it true that if R is an excellent domain, $c \neq 0$ is in R, and R_c is regular, then c has a power that is a completely stable test element?

This is known if R is F-finite or essentially of finite type over an excellent semilocal ring. It is also known if $\dim(R) \leq 2$. It is not known in general in the excellent case, even if R is a homomorphic image of a regular ring.

Some references

Bibliography

[Ho94] M. Hochster, *Tight closure in equal characteristic, big Cohen-Macaulay algebras, and solid closure* in *Commutative Algebra: Syzygies, Multiplicities and Birational Algebra*, Contemp. Math. **159** Amer Math Soc., Providence, R. I., 1994,173–196.

[HH89] M. Hochster and C. Huneke, *Tight closure and strong F-regularity*, Mémoires de la Société Mathématique de France, numéro **38**, 1989,119–133.

[HH90] M. Hochster and C. Huneke, *Tight closure, invariant theory, and the Briançon-Skoda theorem*, J. Amer. Math Soc. **3** (1990) 31–116.

[HH] M. Hochster and C. Huneke, *Tight closure in characteristic 0*, preprint. A version is available at http://www.math.lsa.umich.edu/∼hochster/msr.html.

[Hu96] C. Huneke, *Tight Closure and Its Applications*, Proc. of the C.B.M.S. Conference held at Fargo, North Dakota, July, 1995, C.B.M.S. Regional Conference Series, Amer. Math. Soc., Providence, R.I., 1996.

[Jia21] Z. Jiang, *Test elements in equal characteristic semianalytic algebras*, preprint, arXiv:2104.12867v1 [math.AC].

Thanks for listening!