THIRTEEN OPEN
QUESTIONS IN
COMMUTATIVE ALGEBRA

Speaker: Mel Hochster

This is an edited version of the slides used at a talk at a conference in honor of Joseph Lipman on the occasion of his sixty-fifth birthday at a conference at Purdue University in July, 2004. Each question is followed by some very terse comments on its status and some references, which are sometimes woven into the comments and sometimes follow them. The references given are, in most cases, only a small sample of the relevant literature.

1. The Zariski-Lipman conjecture: Let R be a finitely generated $\mathbb{C}$-algebra and P a prime of R. If $\mathrm{Der}_{\mathbb{C}}(R,\, R)_P$ is R_P-free, then R_P is regular.

Comments: R_P is normal (Lipman). The hypersurface case is known (Scheja-Storch). The graded case is known (Hochster). The dimension 2 case implies the general case (Flenner), but is open even for complete intersections in $\mathbb{C}^4$.

J. Lipman, *Free derivation modules on algebraic varieties*, Amer. J. Math. **87** (1965) 874–898.

G. Scheja and U. Storch, *Über differentielle Abhängigkeit bei idealen analytischer Algebren*, Math. Z. **114** (1970) 101–112.

____, *Differentielle Eigenschaften des Lokaliserungen analytischer Algebren*, Math. Ann. **197** (1972) 137–170.

M. Hochster, *The Zariski-Lipman conjecture in the graded case*, J. of Alg. **47** (1977) 411–424.

H. Flenner, *Extendability of differential forms on nonisolated singularities*, Invent. Math. **94** (1988) 317–326.

2. Resolution of singularities: Given a reduced and irreducible excellent scheme, or simply a domain finitely generated over a field, over $\mathbb{Z}$, or over an excellent discrete valuation ring, one wants to find a proper birational map of a scheme or variety onto it whose local rings are regular. Better yet, if R is a domain, one wants $f_1, \ldots, f_n \in R$ such that $V(f_1, \ldots, f_n)$ is the singular locus and, for all i, $R[f_1/f_i, f_2/f_i, \ldots, f_n/f_i]$ is regular. This is a local algebraic version of *blowing up*. Iterations of this process have been the main tool in resolving singularities.

Comments: The literature is HUGE. There are a handful of references below. The recent exposition by H. Hauser listed below contains many more. Hauser calls Zariski "the grandfather" of resolution of singularities. Much notable work was done by S. S. Abhyankar. Hironaka solved the problem in equal characteristic zero. Lipman gave an optimal treatment for surfaces. Recent work of De Jong solves the problem in characteristic p and mixed characteristic if the birationality condition is relaxed and one allows a finite field extension of function fields.

S. S. Abhyankar, *Local uniformization of algebraic surfaces over ground fields of characteristic* $p \neq 0$, Ann. Math. **63** (1956) 491–526.

____, *Resolution of singularities of embedded algebraic surfaces*, Academic Press, New York, 1966.

H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero* I, II. Ann. of Math. (2) **79** (1964) 109–203; 205–326.

J. Lipman, *Desingularization of 2-dimensional schemes*, Ann. Math. **107** (1978) 151–207.

E. Bierstone and P. D. Milman, *Uniformization of analytic spaces*, J. Amer. Math. Soc. **2** (1989) 801–836. A. J. de Jong, *Smoothness, semi-stability and alterations*, Publ. I.H.E.S. Math. No. **83** (1996) 51–93.

A. J. de Jong, *Families of curves and alterations*, Ann. Inst. Fourier (Grenoble) **47** (1997) 599–621.

H. Hauser, *The Hironaka theorem on resolution of singularities (Or: A proof we always wanted to understand)*, Bull. A.M.S. **40** (2003) 323–403.

3. Cancellation of indeterminates (Zariski problem): If $\mathbb{C}[x_1,\, \ldots,\, x_{n+m}] \cong R[y_1,\, \ldots,\, y_m]$ is $R \cong \mathbb{C}[x_1,\, \ldots,\, x_n]$?

Comments: The answer is yes for $n = 1$ or $n = 2$ (Fujita, Miyanishi, Sugie: cf. [T. Fujita, *On Zariski problem*, Proc. Japan Acad. Ser. A Math. Sci **55** (1979) 106–110], M. Miyanishi and T. Sugie, *Affine surfaces containing cylinderlike* *open sets*, J. Math. Kyoto Univ. **20** (1980) 11–42], and [T. Sugie, *Characterization of the affine plane and the affine threespace*, in *Topological Methods in Algebraic Transformation Groups* (editors H. Kraft, T. Petrie, and G. W. Schwarz), Progress in Math. **80**, Birkhäuser, Boston-Basel-Berlin, 1989, 177–190].)

Be wary about cancellation!

Let R, S be $\mathbb{C}[x,y,z]/\big(xy-(1-z^2)\big)$, $\mathbb{C}[x,y,z]/\big(x^2y-(1-z^2)\big)$. $R \not\cong S$ but $R[t] \cong S[t]$! (*Danielewski surfaces.*) Cf. [W. Danielewski, *On the cancellation problem and automorphism groups of affine algebraic varieties*, preprint, Warsaw, 1989], [K.-H. Fieseler, *On complex affine surfaces with* $\mathbf{C}^+$*-action*, Comment. Math. Helv. **69** (1994) 5–27].

4. The Jacobian conjecture: If $R = \mathbb{C}[x_1, \ldots, x_n]$, and $f : R \to R$ with $f(x_i) = f_i$ for $1 \le i \le n$ is such that $\det(\partial f_i/\partial x_j) \in \mathbb{C} - \{0\}$, then f is an automorphism.

Comments: There have been *many* incorrect proofs: several have been published. Cf. [H. Bass, E. Connell, and D. Wright, *The Jacobian Conjecture: reduction of degree and formal expansion of the inverse*, Bull. A.M.S. **7** (1982) 287–330]. The question remains open even if $n = 2$. A survey is given in [A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), 55–81, Sémin. Congr., 2, Soc. Math. France, Paris, 1997.]

5. Set-theoretic definition: Is there a method for determining the least number of generators of an ideal up to radicals (how many equations are needed to define an algebraic set)?
This seems incredibly difficult. It is not known whether every irreducible curve in $\mathbb{A}^3_{\mathbb{C}}$ is even locally a set-theoretic complete intersection! It is not known whether, for example, the parametric Moh curve

$$x = t^6 + t^{31}, y = t^8, z = t^{10}$$

is a set-theoretic complete intersection at the origin! Cf. [G. Lyubeznik, *A survey of problems and results on the number of defining equations*, in *Commutative Algebra* M.S.R.I. Publ. **15** Springer-Verlag, New York, 1989, 375–390].

6. The strong intersection conjecture: Let (R, m) be a local (Noetherian) ring and let M, $N \neq 0$ be finitely generated R-modules such that $\mathrm{pd}_R M < \infty$ and $M \otimes_R N$ has finite length. Let $I = \mathrm{Ann}_R M$. Then $\dim(N) \le \mathrm{depth}_I R$.

Comments: Cf. [C. Peskine and L. Szpiro, Dimension projective finie et cohomologie locale, Publ. I.H.E.S. **42**, 1973].Many other questions considered there have been settled, including conjectures of H. Bass and M. Auslander. In equal characteristic several follow from the existence of big Cohen-Macaulay modules. In mixed characteristic, others were settled by P. Roberts using intersection theory. M. Auslander's rigidity conjecture was settled negatively by R. Heitmann. See also:

M. Hochster, *Topics in the homological theory of modules over commutative rings*, C.B.M.S. Regional Conf. Ser. in Math. No. **24**, Amer. Math. Soc., Providence, R.I., 1975.

P. Roberts, *Le théorème d'intersection*, C.R. Acad. Sc. Paris Sér. I **304** (1987) 177–180.

P. Roberts, *Multiplicities and Chern classes in local algebra*, Cambridge Tracts in Mathematics 133, Cambridge Univ. Press, Cambridge, England, 1998.

R. Heitmann, *A counterexample to the rigidity conjecture for rings*, Bull. A.M.S. New Series **29** (1993) 94–97.

7. The Buchsbaum-Eisenbud-Horrocks problem: If R is regular local, $\dim(R) = n$, and $\ell(M) < \infty$, then the j th Betti number of M is $\ge \binom{n}{j}$.

Comments: See D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977) 447–485 and R. Hartshorne, *Algebraic vector bundles on projective spaces: a problem list*, Topology **18** (1979) 117–128. This is known if $\dim(R) \le 4$. Weaker form: the sum of the Betti numbers is at least 2^n. Stronger form: the torsion-free rank of a minimal j th module of syzygies is $\ge \binom{n-1}{j-1}$, $1 \le j \le n$. Closely related: if $\ell(M) < \infty$ over (R, m), a local ring, and $\underline{x} = x_1, \ldots, x_n \in m$, then $\ell\big(H_j(\underline{x};\, M)\big) \ge \binom{n}{j}$.

Another somewhat related question: if I, $J \subseteq R$ with (R, m) regular local, $\dim(R) = n$, and $I + J$ m-primary then $\mathrm{Tor}_1^R(R/I, R/J)$ needs $\ge n - \dim(R/I) - \dim(R/J)$ generators. (This is open for $K[[x, y, z]]$ even if I, J are m-primary: one expects $(I \cap J)/IJ$ to need at least three generators.)

The original problem asks for a lower bound for the dimension of $\mathrm{Tor}_j^R(M,K)$. The second asks for a lower bound for the dimension of $\mathrm{Tor}_0^R(\mathrm{Tor}_1^R(R/I,R/J),K)$. One may ask, more generally, for lower bounds for the dimensions of more complicated iterated Tor modules under various conditions on the input modules. Why make it harder? Sometimes, the "harder" question is easier.

Another closely related question: Let (A,m) be an Artin local ring and let $x_1,\,\dots,x_n \in m$. Is the number of generators of $H_1(x_1,\,\dots,x_n;A)$ at least n? This is true if $n=1$ and $n=2$ (S. Dutta).

8. Positivity of Serre multiplicities: Let (R,m) be regular local, $\dim(R)=n$. Let M, N be Noetherian R-modules with $\ell(M\otimes_R N)<\infty$. If $\chi(M,N)=\sum_{j=1}^n \ell\big(\mathrm{Tor}_j^R(M,N)\big)$ then:
(1) $\dim M+\dim N\le \dim R$, (2) $\chi(M,N)\ge 0$, and (3) $\chi(M,N)>0$ iff equality holds in (1).

Comments: This was proved by Serre if R contains a field or if $\widehat{R}$ is formal power series over a DVR. Serre proved (1) in general. Cf. [J.-P. Serre, Algèbre local · Multiplicités, Springer-Verlag Lecture Notes in Math. **11**, Springer-Verlag, New York, 1961. That χ vanishes when it should was proved in [P. Roberts, *The vanishing of intersection multiplicities of perfect complexes*, Bull. A.M.S. **13** (1985) 127–130] and [H. Gillet and C. Soulé, *K-théorie et nullité des multiplicités d'intersection*, C. R. Acad. Sci. Paris Série I no. 3 t. **300** (1985) 71–74]. (2) was proved by O. Gabber: an exposition is given in [P. Berthelot, *Altérations de variétes algébriques [d'après A. J. de Jong]*, Séminaire BOURBAKI, 48ème année, n$^{\mathrm{O}}$ 815, 815-01 – 815-39] using De Jong's alterations. Positivity of multiplicities remains a frustrating open question. It is implied by the existence of small Cohen-Macaulay modules, considered in 12. below.

9. Direct summands of regular rings are Cohen-Macaulay: If R is a direct summand as an R-module of a regular ring S, then must R be Cohen-Macaulay?

This is easy if S is module-finite over R. It is known in equal char.: cf. [M. Hochster and J. L. Roberts, *Rings of invariants of reductive groups acting on regular rings are Cohen-Macaulay*, Advances in Math. **13** (1974) 115–175], [M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*, Advances in Math. **113** (1995), 45–117], and [____, *Tight closure in equal characteristic zero*, Springer-Verlag, to appear]. The result for finitely generated $\mathbb{C}$-algebras is sharpened in [J.-F. Boutot, *Singularités rationelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987) 65–68]: if S has rational singularities and R is a direct summand, then R has rational singularities. The question is open in mixed characteristic. It would follow from a solution of the fourth form of the problem considered in 12. or from a solution of 13.

10. The direct summand and canonical element conjectures; understanding superheight: Let R be regular and $R\subseteq S$ module-finite. Is R a direct summand of S?

This is trivial if $R\supseteq\mathbb{Q}$ (and then R need only be normal), and was proved in char. $p>0$ in [M. Hochster, *Contracted ideals from integral extensions of regular rings*, Nagoya Math. J. **51** (1973) 25–43]. It is easy in dimension ≤ 2. It reduces to the case of formal power series over a DVR in which p generates the maximal ideal. In [R. C. Heitmann, *The direct summand conjecture in dimension three*, Annals of Math. (2) **156** (2002) 695–712] the result was finally proved in mixed characteristic in dimension 3. The direct summand conjecture remains open if $\dim(R)\ge 4$. An equivalent conjecture (although it "appears" stronger) is that: if (R,m,K) is local and $\underline{x}=x_1,\,\dots,x_n$ is a system of parameters, if one maps the Koszul complex $\mathcal{K}_\bullet=\mathcal{K}_\bullet(\underline{x};R)$ to a free minimal resolution $G_\bullet$ of K beginning with $\mathbf{1_R}$, then the induced map $\mathcal{K}_n=R\to \mathrm{syz}^{\,n}K$ is such that the image of 1 is not in $(\underline{x})\mathrm{syz}^{\,n}K$. Letting the system of parameters vary, one sees that one has a (conjecturally nonzero) canonical element in $H_m^n(\mathrm{syz}^{\,n}K)$, or, as pointed out to me by Joe Lipman, a (conjecturally, nonzero) canonical element in $\mathrm{Tor}_n^R\big(K,H_m^n(R)\big)$. Using that the direct summand conjecture implies the canonical element conjecture, one sees that it implies many of the early local homological conjectures (known from Roberts' work). Cf. [M. Hochster, *Canonical elements in local cohomology modules and the direct summand conjecture*, J. of Algebra **84** (1983) 503–553].

The direct summand conjecture is also equivalent to the statement that if $x_1,\,\dots,x_n$ is a system of parameters of a local ring (R,m), then $(x_1\cdots x_n)^t\notin(x_1^{t+1},\dots,x_n^{t+1})R$ (the *monomial conjecture*). Another equivalent statement is this: if (V,x_1V) is a complete DVR and $T=V[[x_2,...,x_n,\,y_1,\,\dots,y_n]]$, and $f=(x_1\cdots x_n)^t-\sum_{j=1}^n y_jx_j^t$, then, if $R=T/fT$, for any map $R\to S$, $\mathrm{ht}\,(x_1,\,\dots,x_n)S$, if finite, is $\le n-1$. More generally, one may ask: can one tell what the largest height a proper expansion of an ideal I of a Noetherian ring R to a Noetherian R-algebra S can achieve? I.e., what is the *superheight* of I? The problem remains unsolved even in simple special cases in low dimension.

11. Localization of tight closure; tight closure vs. plus closure: If R is an excellent Noetherian domain, char. $p > 0$, does the tight closure I^* of I commute with localization? Is it the same as the elements in $IR^+ \cap I$, where R^+ is the integral closure of R in an algebraic closure of its fraction field?

Comments. Recall that $u \in I^*$ if for some $c \neq 0$, $cu^{p^e} \in I^{[p^e]}$ $\forall e \gg 0$, where $I^{[p^e]} = (i^{p^e} : i \in I)R$.

$IR^+ \cap R \subseteq I^*$. By [K. E. Smith,*Tight closure of parameter ideals*, Invent. Math. **115** (1994) 41–60] they are equal if IR_P is generated by part of a system of parameters in R_P for all $P \supseteq I$. Brenner [H. Brenner, *Tight closure and plus closure for cones over elliptic curves*, preprint], [____, *Tight closure and plus closure in dimension two*, preprint] has done the case of homogeneous ideals in the homogeneous coordinate rings of elliptic curves and of other nonsingular curves if the base field is the algebraic closure of $\mathbb{Z}/p\mathbb{Z}$.

Plus closure commutes with localization, and so the second question would solve the localization problem for tight closure.

12. Existence of (big) Cohen-Macaulay (C-M) modules and algebras. Let (R, m, K) be local. We shall say here that M is a *big C-M module or algebra* if $mM \neq M$ and every system of parameters R is a regular sequence on M. We say that M is a (small) C-M module if it is finitely generated.

(1) Does every complete local domain have a small C-M module?

(2) Do there exist big C-M modules?

(3) Do there exist big C-M algebras?

(4) Given a local map of complete local domains $R \to S$ do there exist big C-M algebras B_R, B_S and an R-algebra map $B_R \to B_S$ (weakly functorial big C-M algebras).

(5) Does a version of tight closure theory exist in mixed characteristic?

Comments: both (2), and (3) reduce to the case of complete (even normal) local domains. Small C-M modules are known to exist in dimension 2 and for finitely generated $\mathbb{N}$-graded domains R of char. $p > 0$ with R_0 a perfect field if the ring has an isolated non-C-M point at the origin: this was first observed by R. Hartshorne. The argument is given in [M. Hochster, *Big Cohen-Macaulay modulesand algebras and embeddability in rings of Witt vectors*, in *Proceedings of the Queen's University Commutative Algebra Conference*, Queen's Papers in Pure and Applied Math., No. **42**, 1975, 106–195].

Big C-M modules were shown to exist in equal char. (using reduction to char. p) in [____, *Topics in the Homological Theory of Modules over Commutative Rings*, Proceedings of the Nebraska Regional C.B.M.S. Conference, (Lincoln, Nebraska, 1974), A.M.S., Providence, 1975]. (4) was proved in char. p¿0 in [____and C. Huneke, *Infinite integral extensions and big Cohen-Macaulay algebras*, Annals of Math. **135** (1992) 53–89]: R^+ works, and in char. 0 in the sequel [____, *Applications of the existence of big Cohen-Macaulay algebras*, Advances in Math. **113** (1995) 45–117]. But (4) is also true for two rings of mixed char. in dimension ≤ 3. This is shown, making use of Heitmann's results, in [M. Hochster, *Big Cohen-Macaulay algebras in dimension three via Heitmann's theorem*, J. of Alg. **254** (2002) 395–408]. I have heuristic reasons for believing that the problem of proving (4) in mixed characteristic in general and the problem of developing an analogue of tight closure in mixed characteristic should be nearly equivalent.

13. The vanishing conjecture for maps of Tor and the strong direct summand conjecture: If $A \subseteq R \to S$ are Noetherian rings with A, S regular and R is module-finite and torsion-free over A, the $\mathrm{Tor}_i^A(M, R) \to \mathrm{Tor}_i^R(M, S)$ is 0 for all A-modules M and all $i \geq 1$.

Comments: This can be proved in equal characteristic using either tight closure theory or the weakly functorial existence of big C-M algebras. This powerful result implies both the direct summand conjecture (when S is a field) and the conjecture that direct summands of regular rings are Cohen-Macaulay (when (A, M, K) is regular local $M = K$, and $i = 1$). Cf. [M. Hochster and C. Huneke, *Applications of the existence of big Cohen-Macaulay algebras*, Advances in Math. **113** (1995) 45–117]. f In mixed characteristic, it is a conjecture.

In [N. Ranganathan, *Splitting in module-finite extension rings and the vanishing conjecture for maps of Tor*, Thesis, University of Michigan, 2000] the surprising result is obtained that 13. is equivalent to the following *strong direct summand conjecture*: let (R, m, K) be a regular local ring in which $x \in m - m^2$, let S be a module finite extension domain, and let Q be a height one prime of S lying over xR. Then $xR \subseteq Q$ splits as a map of R-modules (which implies that R splits from S). There is no easy proof known in equal char. 0, and the result is open in mixed characteristic.

Here is one **not-so-open question:** In [Y. Kamoi, K. Kurano, *On maps of Grothendieck groups induced by completion*, J. Alg. **254** (2002) 21–43] it's shown that for a local ring R, $G_0(R) \to G_0(\widehat{R})$ is injective if R is excellent and either Henselian, or has an isolated singularity, or is the local ring of an $\mathbb{N}$-graded algebra of finite type over a field. The authors didn't know an example of non-injectivity.

Here's an example: let $T = K[x, y, u, v]_m$ and $R = T/fT$ where $f = xy - ux^2 - yv^2$. f factors in $\widehat{T}$ and $P = (x, y)R$ becomes principal mod either of the the two minimal primes of $\widehat{R}$. But $[R/(x, y)] \neq 0$ (it is 2-torsion) in $G_0(R)$, even if one maps to $\widehat{R_P}$. If you are sad that this is not open here's something to think about: I don't know a counter-example if R is normal (conjecture: there is one).

Homework: settle one.
Due: May 18, 2009

•Zariski-Lipman conjecture
•Resolution of singularities
•Cancellation of indeterminates
•Jacobian conjecture
•Set-theoretic definition
•Strong intersection
•Buchsbaum-Eisenbud-Horrocks problem
•Serreconjecture on positivity of multiplicities
•Direct summand of regular is Cohen-Macaulay
•Direct summand conjecture
•Tight closure commutes with localization
•Existence of big and/or small Cohen-Macaulay algebras or modules
•Vanishing of maps of Tor