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Faithfulness of top local cohomology modules in domains

Melvin Hochster[1] and Jack Jeffries[2]

We study the conditions under which the highest nonvanishing local cohomology module of a domain R with support in an ideal I is faithful over R, i.e., which guarantee that $H^c_I(R)$ is faithful, where c is the cohomological dimension of I. In particular, we prove that this is true for the case of positive prime characteristic when c is the number of generators of I.

1. Introduction

Throughout, all rings are commutative, Noetherian, associative with identity, and *local* ring $(R,\, \mathfrak{m},\, K)$ means Noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue class field $R/\mathfrak{m} = K$.

The *local cohomology* functors with support in an ideal I of R are defined as $H^i_I(-) := \varinjlim_n \operatorname{Ext}^i_R(R/I^n, -)$. The vanishing or nonvanishing of the modules $H^i_I(R)$ is related to many other interesting algebraic and geometric properties of R and I. For example,

- The least i for which $H^i_I(R) \neq 0$ is the depth of I on R [8, Theorem 9.1];
- The largest i for which $H^i_{\mathfrak{m}}(R) \neq 0$ in a local ring $(R, \mathfrak{m})$ is the dimension of R [8, Theorem 9.3];
- If $(R, \mathfrak{m})$ is a complete local domain, and $H^i_I(R) = 0$ for $i > \dim(R) - 2$, then $\operatorname{Spec}(R) \smallsetminus V(I)$ is connected [8, Theorem 15.11];
- For $R = \mathbb{C}[x_1, \dots, x_n]$, if $H^i_I(R) = 0$ for all $i > t$, then we have that, for all $i > t$, $H^{n+i}((\mathbb{C}^n \smallsetminus V(I))^{\mathrm{an}}) = 0$, where $\_^{\mathrm{an}}$ denotes the associated analytic space [8, Theorem 19.25];

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- The *cohomological dimension of* I, the largest i for which $H_I^i(R) \neq 0$, is a lower bound for the *arithmetic rank* of I, the minimal number of generators of I up to radical [8, Proposition 9.12].

Related to the question of vanishing is the study of annihilators of local cohomology. In this paper, we study the following question, which we state in two equivalent forms:

Question 1.1. (a) If R is a domain that contains a field, I is an ideal of R, and c the cohomological dimension of I, must $H_I^c(R)$ be a faithful R-module?

(b) If R contains a field, I is an ideal of R, and c the cohomological dimension of I, must the annihilator of $H_I^c(R)$ have height zero?

This question is inspired by a conjecture of Lynch [10, 11], which posits that if c is the cohomological dimension of the ideal I of a local ring $(R, \mathfrak{m}, K)$, and J is the annihilator of the local cohomology module $H_I^c(R)$, then R/J has the same Krull dimension as R. A number of positive results on Lynch's conjecture, that is cases where Question 1.1 has an affirmative answer, have been established, including that it holds for rings of dimension at most three. We refer the reader to [2] for a summary of some of these results.

However, Lynch's conjecture is false. The first counterexample to this was given by Bahmanpour [1]; this example is a nonequidimensional algebra over a field of arbitrary characteristic. A nonequidimensional counterexample in dimension three appears in [15]. A counterexample to Lynch's conjecture in a power series ring over a DVR of mixed characteristic is given in forthcoming work of Datta, Switala, and Zhang [3]. We note here that for regular rings of characteristic zero [12] and for strongly F-regular rings of positive characteristic [2] (hence for all regular rings containing a field), every nonzero local cohomology module is faithful.

In this note, we answer Question 1.1 affirmatively in two main cases:

(i) $\operatorname{char}(R) = p > 0$, and $\operatorname{cd}(I) = \operatorname{ara}(I)$;

(ii) R is pure in a regular ring containing a field.

Notably, local cohomology modules in case (i) above have closed support [9], but may have infinitely many associated primes [14].

We also show that cases where there is an affirmative answer to Question 1.1 imply a persistence property for cohomological dimension; see §3 and, in particular, Corollary 3.3.

2. Main results

For an ideal I in a ring T and a T-module M, we denote the cohomological dimension of I with support in M as

$$\operatorname{cd}(I, M) := \sup\{n \in \mathbb{N} \mid H^n_I(M) \neq 0\}.$$

To prepare for the proof of the main theorem, we record a couple of lemmas that are likely known to experts.

Lemma 2.1. *Let T be a Noetherian ring, and I be an ideal.*

$$\begin{aligned}\operatorname{cd}(I, T) &= \max\{\operatorname{cd}(I, M) \mid M \textit{ is an } R\textit{-module}\}\\ &= \max\{\operatorname{cd}(I, T/Q) \mid Q \in \operatorname{Min}(T)\}.\end{aligned}$$

Proof. The first equality is standard; see [8, Theorem 9.6]. For the second, let $c = \operatorname{cd}(I, T)$. By the first equality, we have $\operatorname{cd}(I, T) \geq \operatorname{cd}(I, T/P)$ for all $P \in \operatorname{Spec}(R)$, and $\operatorname{cd}(I, T/P) \geq \operatorname{cd}(I, T/Q)$ if $P \subseteq Q$. Take a prime filtration $\{T_i\}$ of T. From the long exact sequence and the first equality we get right-exact sequences

$$H^c_I(T_i) \to H^c_I(T_{i+1}) \to H^c_I(T/Q_i) \to 0, \qquad Q_i \in \operatorname{Spec}(R),$$

for each i. If $H^c_I(T/P) = 0$ for every minimal prime of T, then $H^c_I(T/Q_i) = 0$ for all i, and inductively we find that $H^c_I(T) = 0$, a contradiction. □

The equivalence of the statements (a) and (b) of Question 1.1 follows easily from the previous lemma.

Proposition 2.2. *Question 1.1 (a) and Question 1.1 (b) are equivalent.*

Proof. If (b) has an affirmative answer, then clearly (a) does as well, since the only height zero ideal in a domain is the zero ideal. If (a) has an affirmative answer, let $c = \operatorname{cd}(I, R)$, and let $r \in \operatorname{Ann}_R(H^c_I(R))$. Then by Lemma 2.1, we have $H^c_I(R/P) \neq 0$ for some $P \in \operatorname{Min}(R)$, and the image of r in R/P annihilates $H^c_I(R) \otimes_R R/P \cong H^c_I(R/P)$. Then, by assumption, the image of r is zero in R/P, so $\operatorname{Ann}_R(H^c_I(R)) \subseteq P$, and consequently the annihilator has height zero. □

See also [1, Proposition 3.1] for another equivalent version of the conjecture. The following lemma is a form of local duality. Note that we are not restricting to finitely generated modules in the statement below.

Lemma 2.3. *Let $(A, \mathfrak{m}, k)$ be a complete Gorenstein local ring of dimension d. Let $E = H^d_{\mathfrak{m}}(A)$, which is an injective hull of $k = A/\mathfrak{m}$ over A, and let $(-)^\vee = \operatorname{Hom}_A(-, E)$ be the Matlis duality functor.*

Then, there is a natural isomorphism $\operatorname{Ext}^i_A(M, A) \cong H^{d-i}_{\mathfrak{m}}(M)^\vee$ for all A-modules M and all $i = 0, \ldots, d$.

Proof. First, we recall that if A is Gorenstein, then the Čech complex shifted by d gives a flat resolution of $H^d_{\mathfrak{m}}(A) \cong E$. Using this to compute Tor gives isomorphisms $H^{d-i}_{\mathfrak{m}}(M) \cong \operatorname{Tor}^A_i(M, E)$. Applying Matlis duality yields

$$H^{d-i}_{\mathfrak{m}}(M)^\vee \cong \operatorname{Tor}^A_i(M, E)^\vee \cong \operatorname{Ext}^i_A(M, E^\vee) \cong \operatorname{Ext}^i_A(M, A),$$

where the second isomorphism is [5, Example 3.6]. □

Lemma 2.4. *Let $(A, \mathfrak{m}_A) \to (T, \mathfrak{m}_T)$ be a local homomorphism of complete local domains. Assume that A is Gorenstein of dimension d. Then,*

$$\operatorname{Ann}_T(H^d_{\mathfrak{m}_A}(T)) = \bigcap_{\phi \in \operatorname{Hom}_A(T,A)} \operatorname{Ker}(\phi).$$

Proof. By Lemma 2.3, there is an isomorphism $\operatorname{Hom}_A(T, A) \cong H^d_{\mathfrak{m}_A}(T)^\vee$, where $(-)^\vee = \operatorname{Hom}_A(-, E_A(A/\mathfrak{m}_A))$ is Matlis duality for A-modules. By faithful exactness of the functor $(-)^\vee$, the map induced by multiplication by $t \in T$ annihilates $\operatorname{Hom}_A(T, A)$ if and only if it annihilates $H^d_{m_A}(T)$. Now,

$$\operatorname{Ann}_T(\operatorname{Hom}_A(T, A)) = \{t \in T \mid \forall \phi \in \operatorname{Hom}_A(T, A),\ \phi(tT) = 0\}.$$

If $\phi(t) = 0$ for all $\phi \in \operatorname{Hom}_A(T, A)$, then $(\phi \circ \cdot t')(t) = \phi(tt') = 0$ for all $t' \in T$, so $\phi(tT) = 0$ as well. The stated equality follows. □

Lemma 2.5. *Let R be a Noetherian domain, $I \subseteq R$ an ideal of R such that $\operatorname{cd}(I, R) = \operatorname{ara}(I)$, and denote this value by c. Suppose that $H^c_I(R)$ is not faithful. Then there is is an injective homorphism $R \to S$, where S is a complete local domain with algebraically closed residue class field, such that $H^c_{IS}(S) \neq 0$ and is not faithful over S.*

Moreover, if R is equicharacteristic, we may choose a coefficient field $K \subseteq S$, we may choose $f_1, \ldots, f_c \in IS$ that generate I up to radicals, and we *may map the formal power series ring* $A := K[\![x_1, \ldots, x_c]\!]$ *continuously to* S *so that the map on* K *is its inclusion in* S *as coefficient field and* $x_i \mapsto f_i$. *This map is automatically injective, and we may identify* $K[\![x_1, \ldots, x_c]\!]$ *with its image* $K[\![f_1, \ldots, f_c]\!] \subseteq S$. *Once this identification is made, we have that* $(x_1, \ldots, x_c)S$ *is an ideal of cohomlogical dimension* c *in* S, *while* $H^c_{(x_1, \ldots, x_c)}(S)$ *is not a faithful* S*-module.*

Proof. Suppose that there is some ideal $I = (f_1, \ldots, f_c)$ such that $H := H^c_I(R) \neq 0$, and there is some $x \neq 0$ such that $xH = 0$. We can localize at a minimal prime ideal in the support of H, which necessarily contains x, to obtain a local choice of S with $R \subseteq S$. We can then complete at the maximal ideal of S: by faithful flatness, we have $H^c_{I\widehat{S}}(\widehat{S}) \cong H^c_I(\widehat{S}) \cong H^c_I(R) \otimes_S \widehat{S} \neq 0$, and the image of x annihilates this module. Note also that since all elements of S are nonzerodivisors on $\widehat{S}$, and so do not lie in any of its associated primes. The completion $\widehat{S}$ might no longer be a domain, but by Lemma 2.1 above, for some $Q \in \mathrm{Min}(\widehat{S})$, we have $H^c_I(\widehat{S}/Q) \neq 0$, R injects into S, which injects into $\widehat{S}/Q$, and x annihilates $H^c_I(\widehat{S}/Q)$. Consequently, $\widehat{S}/Q$ is a new choice of S that is a complete local domain.

Now assume that S is a complete local domain. Fix a coefficient field for S, which we will denote by the same letter K as the residue field. We now want to reduce to the case where K is algebraically closed. We can take a faithfully flat local extension of R with residue field $\overline{K}$; this extension again may not be a domain, but we may pass to the quotient by an associated prime and still have a counterexample, by the same argument as above. Consequently, we have a choice of S that is a complete local domain with algebraically closed residue field K.

Let $(A, \mathfrak{m}_A, K) = (K[\![x_1, \ldots, x_c]\!], (x_1, \ldots, x_c), K)$ be a power series ring over K, and consider the map $\varphi : A \to R$ described in the statement of the lemma. The hypothesis on H implies that φ is injective: otherwise, φ would factor through a local ring $(\overline{A}, \mathfrak{m}_{\overline{A}})$ of dimension less than c, and, by Lemma 2.1,

$$\mathrm{cd}(I, R) = \mathrm{cd}(\mathfrak{m}_{\overline{A}}R, R) = \mathrm{cd}(\mathfrak{m}_{\overline{A}}, R) \leq \mathrm{cd}(\mathfrak{m}_{\overline{A}}, \overline{A}) = \dim(\overline{A}) < c.$$

We may therefore identify the power series ring A with its image in R. $\square$

We are now ready to prove one of our main results. We refer the reader to [4] for basic properties of solid algebras over a domain.

Theorem 2.6. *Let R be a Noetherian domain of characteristic $p > 0$. Let I be an ideal of R such that $\operatorname{cd}(I, R) = \operatorname{ara}(I)$, and denote this value by c. Then,* $\operatorname{Ann}_R(H^c_I(R)) = 0$.

Proof. As in the conclusion of Lemma 2.5, we may assume that R is a complete local domain with algebraically closed residue field K, and that there is a power series subring $A = K[\![x_1, \dots, x_c]\!] \subseteq R$ such that $I = \mathfrak{m}_A R$.

Since $H^c_{\mathfrak{m}_A}(R) \neq 0$, it follows from Lemma 2.4 that $\operatorname{Hom}_A(R, A) \neq 0$; i.e., R is a solid A-algebra. Let J be the annihilator of $H^c_I(R)$. We want to show that $J = 0$; suppose otherwise, to obtain a contradiction.

By definition of J, the intersection of the kernels of the A-linear maps from R/J to A is trivial, so there is an A-linear embedding

$$R/J \hookrightarrow \prod_{\phi \in \operatorname{Hom}_A(R/J, A)} A \qquad\qquad r \mapsto (\phi(r))_\phi.$$

Let P be a minimal prime of J. Then R/P embeds in R/J as an R-module, and so there is an A-linear embedding of R/P into a product of copies of A.

If $q = p^e$ and we replace A by A^q, the inclusion $A^q \to R$ again satisfies the hypotheses of Lemma 2.4, so

$$J = \bigcap_{\phi \in \operatorname{Hom}_A(R,A)} \operatorname{Ker}(\phi) = \bigcap_{\psi \in \operatorname{Hom}_{A^q}(R, A^q)} \operatorname{Ker}(\psi).$$

In particular, there is an A^q-linear embedding of $R/P \hookrightarrow \prod A^q$ into a product of copies of A^q.

Let $J' = JR_P \cap R$ be the P-primary component of J. Choose h such that $(PR_P)^h \subseteq JR_P$, so $P^{(h)}$ is properly contained in J'. Then $P^{(h)}$ cannot contain J, since this would yield a contradiction after localizing at P. Choose $u \in J \smallsetminus P^{(h)}$.

Choose $q = p^e$ so that $P^{[q]} \subseteq P^{(h)}$. Consider $R/P^{(h)}$ as an R^q module. Then the elements of the prime ideal P^q of R^q (i.e., the set of q th powers of elements in P) annihilate $R/P^{(h)}$, and so $R/P^{(h)}$ may be viewed as an $R^q/P^q \cong (R/P)^q$-module. In fact, $R/P^{(h)}$ is a torsion-free $(R/P)^q$-module: if $\overline{r^q} \in (R/P)^q$, $\overline{s} \in R/P^{(h)}$, and $\overline{r^q} \cdot \overline{s} = 0$, then $r^q s \in P^{(h)}$ in R, so either $s \in P^{(h)}$ (so that $\overline{s} = 0$), or else $r^q \in P$ (so that $\overline{r^q} = 0$ is zero) by primariness of $P^{(h)}$. Since R is complete local with an algebraically closed residue field, it is F-finite, and the images of a finite generating set for R as an R^q-module yield a finite generating set for $R/P^{(h)}$ as an $(R/P)^q$-module.

Hence, $R/P^{(h)}$ embeds $(R/P)^q$-linearly in a finitely generated free $(R/P)^q$-module. Consequently, the image v of u in $R/P^{(h)}$ has nonzero coordinate projection in some copy of $(R/P)^q$. The composition $R \twoheadrightarrow R/P^{(h)} \to (R/P)^q$ gives an A^q-linear map such that the image of u is not 0. Since $(R/P)^q$ embeds in a product of copies of A^q, further composition gives an A^q-linear map $R \to A^q$ such that the image of $u \in J$ is nonzero. This is a contradiction. □

Corollary 2.7. *Let R be a Noetherian ring of characteristic $p > 0$, and I an ideal for which $\operatorname{cd}(I, R) = \operatorname{ara}(I) = c$. Then $\operatorname{Ann}_R(H^c_I(R))$ has height zero.*

Proof. This is immediate from Theorem 2.6 and Proposition 2.2. □

We do not know whether the analogues of Theorem 2.6 and Corollary 2.7 hold in equal characteristic zero. By extending results of Huneke, Katz, and Marley [6], we may reduce this to the case of ideals with at most three generators. This is immediate from:

Proposition 2.8. *A local cohomology module $H^n_I(M)$ of a module M over a Noetherian ring R with support in an n-generated ideal for $n \geq 4$ is isomorphic with a local cohomology module $H^3_J(M)$ where J has at most three generators.*

Proof. It suffices to show that if $n \geq 4$, we can reduce the number of generators and the cohomological index by 1. Let u, v, x, y be four of the generators and $\mathfrak{A}$ the ideal generated by the $n-4$ remaining generators of I. Let $J = (xu, yv, xv + yu, \mathfrak{A})R$. It suffices to show that $H^{n-1}_J(M) = H^n_I(M)$. Note that xv, yu are roots of

$$z^2 - (xv + yu)z + (xv)(yu) = 0$$

and so are integral over J and in its radical. It follows that $(u, v)(R/\mathfrak{A}) \cap (x, y)(R/\mathfrak{A})$, which has the same radical as $(u, v)(x, y)(R/\mathfrak{A})$, also has the same radical as $(xu, yv, xv + yu)(R/\mathfrak{A})$. Hence, up to radicals, J is the intersection of $I_1 = (u, v)R + \mathfrak{A}$ and $I_2 = (x, y)R + \mathfrak{A}$. The Mayer-Vietoris sequence for local cohomology then yields

$$\cdots \longrightarrow H^{n-1}_{I_1}(M) \oplus H^{n-1}_{I_2}(M) \longrightarrow H^{n-1}_J(M) \\ \longrightarrow H^n_{I_1+I_2}(M) \longrightarrow H^n_{I_1}(M) \oplus H^n_{I_2}(M) \longrightarrow \cdots$$

and the result follows because $I_1 + I_2 = I$ and the first and last of the four terms shown are 0, since I_1, I_2 both have only $n-2$ generators. □

We now establish another case where Question 1.1 has an affirmative answer.

Theorem 2.9. *Let R be a pure R-submodule of a domain S that is an R-algebra. Let I be an ideal of R, and $c = \mathrm{cd}(I, R)$. If $\mathrm{Ann}_R(H^c_{IS}(S)) = 0$, and, in particular, if $\mathrm{Ann}_S(H^c_{IS}(S)) = 0$ then $\mathrm{Ann}_R(H^c_I(R)) = 0$.*

In particular, the conclusion holds when S is a regular domain containing a field and R is pure in S.

Proof. Since $R \to S$ is pure, it remains injective when we tensor over R with $H^c_I(R)$, and so we have an injection $H^c_I(R) \to H^c_I(S) \cong H^c_{IS}(S)$; therefore, the latter is nonzero. But any nonzero $r \in R$ that kills $H^c_I(R)$ will also kill the nonzero module $H^c_I(S) \cong H^c_I(R) \otimes_R S$, contradicting the hypothesis.

It remains to justify the statements made in the regular case. If K has characteristic $p > 0$, then the theorem then follows from [7, Lemma 2.2]. If K has characteristic 0, we may use the fact that every nonzero local cohomology module with coefficients in S is faithful [12, Corollary 3.6].

Alternatively, both cases follow from the basic theorems of Lyubeznik [12, 13]: if S is regular and contains a field, we may see that any nonzero local cohomology module $M = H^i_J(S)$ of S is faithful as follows. Localize at a minimal prime of the support of M, which produces a nonzero local cohomology module supported only at the maximal ideal of an equicharacteristic regular local ring, and so isomorphic with a nonzero finite direct sum of copies of the injective hull of the residue field, and, consequently, faithful over the localization of S and therefore over S. □

3. Persistence of cohomological dimension

As a consequence of Lemma 2.1, if $R \to S$ is a ring homomorphism, and I is an ideal of R, then $\mathrm{cd}(I, R) \geq \mathrm{cd}(IS, S)$. It is easy to see that equality holds if R is a direct summand of S as an R-module, but the inequality is strict in general. Likewise, the arithmetic rank of an ideal can decrease when passing to a larger algebra. We pose the following conjecture.

Conjecture 3.1. *If R is a Noetherian domain, and S is a solid R-algebra, then for every ideal I of R, we have $\mathrm{cd}(I, R) = \mathrm{cd}(IS, S)$.*

Remark 3.2. Conjecture 3.1 holds under the stronger assumption that S is a module-finite R-algebra. Indeed, let $c = \mathrm{cd}(I, R)$. By Gruson's theorem (see, e.g., [16, Corollary 4.3]), since S is a faithful R-module, $H^c_{IS}(S) \cong S \otimes_R H^c_I(R) \neq 0$, since $H^c_I(R) \neq 0$.

We observe that if Question 1.1 has a positive answer for a ring R and ideal I, then Conjecture 3.1 holds for R, I. Indeed, if $\phi : S \to R$ is a nonzero R-linear map, so that $\phi(s) = r \neq 0$ for some $r \in R, s \in S$, let $\phi' = \phi \circ (\cdot s)$, so that $\phi'|_R = \cdot r$, the map of multiplication by r. Applying the functor $H^c_I(-)$, we obtain that the nonzero map of multiplication by r on $H^c_I(R)$ factors through $H^c_{IS}(S)$, which must then be nonzero.

Thus, as a corollary of Theorem 2.6, we obtain the following special case of Conjecture 3.1.

Corollary 3.3. *If R is a domain of positive characteristic, and I is an ideal such that* $\mathrm{cd}(I, R) = \mathrm{ara}(I, R) = c$*, then for any solid R-algebra S, we have* $\mathrm{cd}(IS, S) = \mathrm{ara}(IS, S) = c$.

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References

[1] K. Bahmanpour, *A note on Lynch's conjecture*, Comm. Algebra **45** (2017), no. 6, 2738–2745.

[2] A. F. Boix and M. Eghbali, *Annihilators of local cohomology modules and simplicity of rings of differential operators*, Beitr. Algebra Geom. **59** (2018), no. 4, 665–684.

[3] R. Datta, N. Switala, and W. Zhang, *Annihilators of D-modules in mixed characteristic*, `arXiv:1907.09948`.

[4] M. Hochster, *Solid closure*, in: Commutative Algebra: Syzygies, Multiplicities and Birational Algebra, Contemp. Math. **159**, Amer. Math. Soc., Providence, RI, (1994), 103–172.

[5] C. Huneke, *Lectures on local cohomology*, Appendix 1 by Amelia Taylor, Contemp. Math. **436**, Interactions Between Homotopy Theory and Algebra, pp. 51–99, Amer. Math. Soc., Providence, RI, (2007).

[6] C. Huneke, D. Katz, and T. Marley, *On the support of local cohomology*, J. Algebra, **322** (2009), no. 9, 3194–3211.

[7] C. Huneke and J. Koh, *Cofiniteness and vanishing of local cohomology modules*, Math. Proc. Cambridge Philos. Soc. **110** (1991), no. 3, 421–429.

[8] S. Iyengar, G. Leuschke, A. Leykin, C. Miller, E. Miller, A. Singh, and U. Walther, 24 Hours of Local Cohomology, Graduate Studies in Mathematics **87**, American Mathematical Society, Providence, RI, (2007).

[9] M. Katzman, *The support of top graded local cohomology modules*, in: Commutative Algebra, pp. 165–174, Lect. Notes Pure Appl. Math. **244**, Chapman & Hall/CRC, Boca Raton, FL, (2006).

[10] L. Lynch, *Annihilators of local cohomology modules*, Thesis, University of Nebraska, (2011).

[11] L. Lynch, *Annihilators of top local cohomology modules*, Comm. Algebra **40** (2012), 542–551.

[12] G. Lyubeznik, *Finiteness properties of local cohomology modules (an application of D-modules to commutative algebra)*, Invent. Math. **113** (1993), no. 1, 41–55.

[13] G. Lyubeznik, *F -modules: applications to local cohomology and D-modules in characteristic* $p > 0$, J. Reine Angew. Math. **491** (1997), 65–130.

[14] A. Singh and I. Swanson, *Associated primes of local cohomology modules and of Frobenius powers*, Int. Math. Res. Not. (2004), no. 33, 1703–1733.

[15] A. Singh and U. Walther, *On a conjecture of Lynch*, `arXiv:1911.01455`.

[16] W. Vasconcelos, *Divisor theory in module categories*, North-Holland Mathematics Studies, No. 14, Notas de Matemática No. 53. North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York, (1974).

Department of Mathematics, University of Michigan
Ann Arbor, MI 48109–1043, USA
E-mail address: hochster@umich.edu

Department of Mathematics, University of Nebraska-Lincoln
Lincoln, NE 68588–0037, USA
E-mail address: jack.jeffries@unl.edu

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