LOCAL COHOMOLOGY AND GROUP ACTIONS

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2011

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To my parents, David and Wendy, my sister, Anne, my brother, David, and Daniel.

ACKNOWLEDGEMENTS

Mel Hochster has provided more mathematical guidance than I could have ever imagined. I am extremely grateful for his time (which I have enjoyed very much) and for sharing invaluable ideas, suggestions, and advice with me. I am also very thankful for his friendship.

I would like to express appreciation for my family, who have put up with many years of my mathematical frustrations and joys. In particular, many thanks to my parents, David and Wendy, to my sister, Anne, and to my brother, David.

I am ever grateful for the support of Daniel Hernández throughout my time here in Ann Arbor. It was serendipitous to have met him (two desks away) during our first year of graduate school, and since then, my life been enriched in many ways.

Very special thanks go to Karen Smith for her generous support and advice, for her family's friendship, for being an amazing role model, for motivating me to study algebra, and for reading my thesis.

I would also like to thank Harm Derksen for answering questions on invariant theory, and the rest of my committee, Wenliang Zhang and Greg Dowd. I also appreciate the support of the NSF grant DMS-0502170.

I am grateful to Paul Sally for mentoring me as an undergraduate, and for steering me to the University of Michigan, and to László Babai, whose courses inspired me and gave me the confidence to pursue mathematics.

Lastly, thanks also go out to the many friends who have made my time at the University of Michigan bright and fulfilling, both mathematically, and just as importantly, non-mathematically. Specifically, I would like thank Gerardo Hernández Dueñas and Yadira Baeza Rodriǵuez (and, of course, Gerardito), Luis Núñez Betancourt, Ben and Alison Weiss, Richard and Laís Vasques, José Gonzalez and Erin Emerson, Luis Serrano, Ryan Kinser, Jasun Gong, Kevin Tucker, Alan Stapledon, Stephanie Jakus, Chelsea Walton, Zhibek Kadyrsizova, Hyosang Kang, and Yogesh More.

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CHAPTER I

Introduction

I.1 History

The focus of this thesis is on understanding local cohomology modules. Given an ideal I of a Noetherian ring R and an R-module M, the *local cohomology modules of M with support in I* are a family of R-modules indexed by nonnegative integers i, and are denoted $H^i_I(M)$. These modules capture many properties of R, I, and M, making them remarkably useful. For example, the first local cohomology module of a ring may be viewed geometrically as the obstruction to extending sections of sheaves from an open set to the whole space. The dimension of a local ring and the depth of a module on an ideal can also be characterized in terms of local cohomology, as can the Cohen-Macaulay and Gorenstein properties.

Understanding the structure of local cohomology modules is both intriguing and challenging. For example, the only nonzero local cohomology module of a polynomial ring R over a field with support in its homogeneous maximal ideal $\mathfrak{m}$ is isomorphic to the smallest injective module containing $R/\mathfrak{m}$ (and gives a concrete realization of this injective hull). Injective modules are rarely finitely generated, and local cohomology modules are often unwieldy and can be difficult to understand.

A major goal of this thesis is to understand local cohomology modules of polynomial rings with support in ideals generated by determinants. More precisely, if

$X = [x_{ij}]$ is an $r \times s$ matrix of indeterminates, where $r \leq s$, consider the polynomial ring R over a field k in the entries of X, i.e.,

$$R = k \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1s} \\ x_{21} & x_{22} & \dots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r1} & x_{r2} & \dots & x_{rs} \end{bmatrix}.$$

The ideals I_t generated by all the $t \times t$ minors of X are examples of *determinantal ideals*, a class of much-studied ideals that provide a rich source of naturally-arising examples in commutative algebra, algebraic geometry, invariant theory, and combinatorics. This thesis is particularly concerned with understanding local cohomology modules of R with support in $I := I_r$, i.e., the ideal generated by the maximal minors of X.

The behavior of the local cohomology modules of the ring with support in the ideal generated by the maximal minors of the matrix X depends strongly on the characteristic of the ring. In prime characteristic, by results of Hochster and Eagon and of Peskine and Szpiro, there is only one nonzero such local cohomology module:

Theorem I.1.1 ([HE71, Theorem 1], [PS73, Théorème III.4.1])**.** *Suppose that X is an $r \times s$ matrix of indeterminates, that k is a field of prime characteristic, that $R = k[X]$ is the polynomial ring over k in the entries of X, and that I is generated by the maximal minors of X. Then the only nonzero local cohomology module of the form $H^i_I(R)$ has index $i = s - r + 1$, the depth of I.*

In characteristic zero, the case on which this thesis is focused, the minimum index for which $H^i_I(R) \neq 0$ is still $i = s - r + 1$, the depth of I [HE71, Theorem 1]. However, an argument of Hochster, Huneke, and Lyubeznik shows that the maximum nonvanishing index is a number almost r times larger:

Theorem I.1.2 ([HL90, Remark 3.13]). *Suppose that X is an $r \times s$ matrix of indeterminates, that k is a field of characteristic zero, that $R = k[X]$ is the polynomial ring over k in the entries of X, and that I is generated by the maximal minors of X. Then*

$$\max\{i \mid H_I^i(R) \neq 0\} = r(s-r)+1.$$

Their proof uses invariant theory and the fact that $r(s-r)+1$ is the dimension of the k-subalgebra of R generated by the maximal minors of X, the homogeneous coordinate ring of the Plücker embedding of the Grassmann variety of r-planes in s-space.

The only previously known explicit description of such a local cohomology module in characteristic zero is due to Walther:

Example I.1.3 ([Wal99, Example 6.1]). Assume that X is a 2×3 matrix of indeterminates, that k is a field of characteristic zero, that $R = k[X]$ is the polynomial ring over k in the entries of X, and that I is generated by the the three 2×2 minors of X. Then $H_I^3(R)$ is isomorphic to the injective hull of k over R.

Walther's example motivates the following question:

Question I.1.4. Suppose that X is an $r \times s$ matrix of indeterminates, that k is a field of characteristic zero, that $R = k[X]$ is the polynomial ring over k in the entries of X, and that I is generated by the maximal minors of X. If $d = r(s-r)+1$ is the "maximum nonvanishing" index noted in Theorem I.1.2, then is $H_I^d(R)$ isomorphic to the injective hull of k over R?

Along with a Gröbner basis algorithm, in the proof of Example I.1.3, Walther employs a powerful theorem of Lyubeznik, proved using the D-module structure of local cohomology modules:

Theorem I.1.5 ([Lyu93, Theorem 3.4]). *Let k be a field of characteristic zero, and let R be a regular k-algebra. If a local cohomology module $H_I^i(R)$ is supported only at* *a maximal ideal $\mathfrak{m}$, then $H^i_I(R)$ is isomorphic to a finite direct sum of copies of the injective hull of $R/\mathfrak{m}$ over R.*

With R, I, and d as in Theorem I.1.4, it can easily be checked that $H^d_I(R)$ is supported only at the homogeneous maximal ideal of R. Thus, by Theorem I.1.5, we know that $H^d_I(R)$ must be isomorphic to the direct sum of a finite number of copies of the injective hull of k over R. Confirming an affirmative answer to to Question I.1.4 is therefore equivalent to proving that the number of copies of the injective hull is *one*.

I.2 Main Results

This thesis answers Question I.1.4 affirmatively: $H^d_I(R)$ is isomorphic to exactly one copy of the injective hull of k over R. Our method relies on invariant theory, as well as the work of Lyubeznik cited earlier. The thesis also provides information about the local cohomology modules $H^i_I(R)$ at indices $i < d$. Our main result regarding the local cohomology modules $H^i_I(R)$ in the characteristic zero case is the following:

Main Theorem on Minors (V.10)**.** *Let k be a field of characteristic zero and let X be an $r \times s$ matrix of indeterminates, where $r < s$. Let $R = k[X]$ be the polynomial ring over k in the entries of X, and let I be its ideal generated by the maximal minors of X. Given an R-module M, let $E_R(M)$ denote the injective hull of M over R.*

(a) *Let $d = \max\{i : H^i_I(R) \neq 0\}$, so that $d = r(s-r)+1$ by Theorem I.1.2. Then*

$$H^d_I(R) \cong E_R(k).$$

(b) *$H^i_I(R) \neq 0$ if and only if $i = (r-t)(s-r)+1$ for some $0 \leq t < r$.*

(c) *Furthermore, if* $i = (r-t)(s-r)+1$*, then*

$$H^i_I(R) \hookrightarrow E_R(R/I_{t+1}) \cong H^i_I(R)_{I_{t+1}},$$

where I_{t+1} *is the ideal of* R *generated by the* $(t+1)\times(t+1)$ *minors of* X *(which is prime by* [HE71, Theorem 1]*). In particular,* $\mathrm{Ass}_R(H^i_I(R)) = \{I_{t+1}(X)\}$.

Note that there is precisely one nonvanishing local cohomology module of the form $H^i_I(R)$ for every possible size minor of X, and that each nonvanishing $H^i_I(R)$ injects into a specific indecomposable injective module. Moreover, this result is proven independently of Theorem I.1.2.

The proof of Main Theorem on Minors V.10 takes advantage of the natural action of the group $G = SL_r(k)$ on the ring R. The fact that this group also acts on each of the local cohomology modules is a powerful tool. A classical result from invariant theory is that R^G, the subring of invariant elements of R, is the k-subalgebra of R generated by the maximal minors of X [Wey39, Theorem 2.6.A]. This means that the ideal I of R generated by the maximal minors of X is the expansion of the homogeneous maximal ideal of R^G to R.

This technique, in fact, can be extended more generally to a polynomial ring with an action of any linearly reductive group. Indeed, we prove the following more general theorem:

Main Theorem (IV.8)**.** *Let* R *be a polynomial ring over a field* k *of characteristic zero with homogeneous maximal ideal* $\mathfrak{m}$*. Let* G *be a linearly reductive group over* k *acting by degree-preserving* k*-automorphisms on* R*, such that* R *is a rational* G*-module (see* Definition II.4.3*). Assume that* $A = R^G$ *has homogeneous maximal ideal* $\mathfrak{m}_A$*, let* $d = \dim A$*, let* $I = \mathfrak{m}_A R$*, and let* $E_R(k)$ *denote the injective hull of* k *over* R*. Then* $H^d_I(R) \neq 0$ *and* I *is generated up to radicals by* d *elements and not fewer, so that* $H^i_I(R) = 0$ *for* $i > d$*. Moreover, the following hold:*

(a) *If* $i < d$*, then* $\mathfrak{m}$ *is not an associated prime of* $H_I^i(R)$*, i.e.,* $H_{\mathfrak{m}}^0(H_I^i(R)) = 0$*.*

If, in addition, $H_I^d(R)$ *is supported only at* $\mathfrak{m}$ *(e.g., this holds if, after localization at any of the indeterminates of* R*,* I *requires fewer than* d *generators up to radical), then*

(b) $V := \operatorname{Soc} H_I^d(R)$ *is a simple* G*-module, and*

(c) *As rational* $R[G]$*-modules* (see Definition II.4.9), $H_I^d(R) \cong E_R(k) \otimes_k V$*.*

I.3 Outline

In Chapter II, we provide background material necessary in understanding the thesis work. Section II.1 focuses on associated primes and support, Section II.2 on local cohomology modules, Section II.3 on determinantal ideals, Section II.4 on linearly reductive groups and G-modules, and Section II.5 on the spectral sequence of a double complex. Chapter III presents additional preliminary definitions and lemmas regarding graded duals and G-modules. Proving the Main Theorem IV.8 is the goal of Chapter IV, and proving the Main Theorem on Minors V.10 is the focus of Chapter V. The thesis concludes with Chapter VI, which provides the proof of Theorem VI.3, a vanishing result on certain iterated local cohomology modules of the form $H_{\mathfrak{m}}^1(H_I^i(R))$. Through a spectral sequence argument, this theorem helps to further describe the local cohomology modules with support in ideals of maximal minors.

CHAPTER II

Background

II.1 Associated Primes and Support

This section reviews the theory of associated primes and support necessary to complete the proofs of the Main Theorem IV.8 and the Main Theorem on Minors V.10. Throughout this section, all rings are assumed to be Noetherian. However, for most results stated, we do not need to assume that modules are finitely generated. We will note when this assumption is needed. Our primary reference is [Mat80].

Definition II.1.1 (Associated prime of a module)**.** Given a Noetherian ring R and an R-module M, an *associated prime of* M is a prime ideal $\mathfrak{p}$ of R such that $R/\mathfrak{p} \hookrightarrow M$ as R-modules, or, equivalently, if $\mathfrak{p} = \operatorname{ann}_R(u)$ for some element $u \in M$. The set of associated primes of M is denoted $\operatorname{Ass}_R(M)$.

Proposition II.1.2. *Given a Noetherian ring R, ideals that are maximal elements of the set $\{\operatorname{ann}_R(x) \mid x \in M, x \neq 0\}$ are associated primes of M.*

Corollary II.1.3. *Given a ring R and an R-module M, the union of the associated primes is the set of all zero divisors on M in R.*

Definition II.1.4. Given a ring R and an R-module M, the *support of* M is the following collection of prime ideals of R: $\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$.

Theorem II.1.5. *Given a ring R and a nonzero R-module M, neither $\operatorname{Ass}_R(M)$ nor $\operatorname{Supp}_R(M)$ is empty. Moreover, $\operatorname{Supp}_R(M) \supseteq \operatorname{Ass}_R(M)$, and the minimal elements of $\operatorname{Ass}_R(M)$ and of $\operatorname{Supp}_R(M)$ are the same.*

Proposition II.1.6. *If R is a ring, $\Sigma \subseteq R$ is a multiplicative system, and M is an R-module, then $\operatorname{Ass}_R(\Sigma^{-1}M) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) \mid \Sigma \cap \mathfrak{p} = \emptyset\}$.*

Theorem II.1.7 ([Mat80, Theorem 12])**.** *If R is a Noetherian ring, M is an R-module, and S is flat Noetherian R-algebra, then*

$$\operatorname{Ass}_R(M \otimes_R S) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \operatorname{Ass}_S(S/\mathfrak{p}S) = \bigcup_{\substack{\mathfrak{p} \in \operatorname{Ass}_R(M) \\ S/\mathfrak{p}S \neq 0}} \mathfrak{p}S.$$

The proof of Theorem II.1.7 extends from the case that M is a finitely-generated module: Since any module is the direct limit of its finitely generated submodules, if for some prime $\mathfrak{p}$, $R/\mathfrak{p}$ injects into M, then it must inject into a finitely-generated submodule of M. The result then follows from the commutativity of direct limits with tensor products.

Proposition II.1.8. *Suppose that R is an $\mathbb{N}$-graded ring, and that M is a $\mathbb{Z}$-graded R-module. Then the associated primes of M are homogeneous.*

A consequence of Proposition II.1.8 is that if R is an $\mathbb{N}$-graded ring with $R_0 = k$, a field, and M is a $\mathbb{Z}$-graded R-module, then every associated prime of M must be contained in the homogeneous maximal ideal of R.

II.2 Local Cohomology Modules

In this section, we will present the material on local cohomology modules that is needed to prove the main results. The main references for this section are [BH93] and [Har67]. Throughout, a *local* ring $(R, \mathfrak{m}, k)$ is a Noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

Definition II.2.1 (Injective module)**.** An R-module E is *injective* if for any injection $N \hookrightarrow M$ of R-modules, the induced map $\operatorname{Hom}_R(M,E) \to \operatorname{Hom}_R(N,E)$ is surjective.

Definition II.2.2 (Injective resolution)**.** If R is a ring and M is an R-module, an *injective resolution* of M is a complex of injective R-modules

$$0 \to E_0 \to E_1 \to E_2 \to \dots$$

such that $M = \ker(E_0 \to E_1)$, and the complex is exact at every other spot.

Definition II.2.3 (Injective dimension)**.** If R is a ring and M is a nonzero R-module, the *injective dimension* of M is the length of the shortest injective resolution of M. If no finite resolution exists, we say that the injective dimension of M is infinite.

Definition II.2.4 (Right derived functor)**.** Given a ring R, suppose that F is a covariant left-exact functor from the category of R-modules to itself. Given an R-module M, take an injective resolution of M:

$$0 \longrightarrow E_0 \xrightarrow{f_0} E_1 \longrightarrow \dots \longrightarrow E_{i-1} \xrightarrow{f_{i-1}} E_i \xrightarrow{f_i} E_{i+1} \longrightarrow \dots$$

Apply F to the resolution to get:

(II.2.4.1)

$$0 \longrightarrow F(E_0) \xrightarrow{F(f_0)} F(E_1) \longrightarrow \dots \longrightarrow F(E_{i-1}) \xrightarrow{F(f_{i-1})} F(E_i) \xrightarrow{F(f_i)} F(E_{i+1}) \longrightarrow \dots$$

The i^{th} *right derived functor of* F, denoted R^iF, is the functor whose value on M is $R^iF(M) = \ker F(f_i) \,/\, \operatorname{Im} F(f_{i-1})$, the i^{th} cohomology module of (II.2.4.1).

Definition II.2.5 (Local cohomology)**.** Let R be a Noetherian ring, let I be an ideal of R, and let M be an R-module. Let Γ be the functor from the category of R-modules to itself such that $\Gamma(M) = \bigcup_{t=1}^{\infty} \operatorname{Ann}_M(I^t) \subseteq M$. The i^{th} *local cohomology of* M *with support in* I is the i^{th} right derived functor of Γ applied to M, $R^i\Gamma(M)$.

There are several equivalent definitions; each definition makes certain properties of local cohomology modules more apparent. The following definition is needed in one such description of local cohomology:

Definition II.2.6 (Ext functor)**.** Suppose that R is a ring and that M and N are R-modules. Then the i^{th} right derived functor of the functor $\operatorname{Hom}_R(M,-)$, applied to N, $R^i \operatorname{Hom}_R(M,-)(N)$, is denoted $\operatorname{Ext}^i_R(M,N)$.

Two particularly useful characterizations of local cohomology are presented in the following theorem:

Theorem II.2.7. *Let R be a Noetherian ring, and M an R-module. Suppose that I is an ideal of R, and that $\sqrt{I} = \sqrt{(f_1,\ldots,f_\mu)}$. Then the following are the isomorphic R-modules:*

(a) $H^i_I(M)$.

(b) $\varinjlim_t \operatorname{Ext}^i_R(I^t, M)$.

(c) *The i^{th} cohomology of the complex:*

$$0 \to M \to \bigoplus_{i=1}^{\mu} M_{f_i} \to \bigoplus_{1\le i<j\le\mu} M_{f_i f_j} \to \ldots \to \bigoplus_{i=1}^{\mu} M_{f_1\ldots\widehat{f_1}\ldots f_\mu} \to M_{f_1\ldots f_\mu} \to 0,$$

where the map on each summand is defined as follows: if $1 \le i_1 < \ldots < i_n \le \mu$, the map

$$M_{f_{i_1}\cdots f_{i_n}} \to \bigoplus_{1\le l_1<\ldots<l_{n+1}\le\mu} M_{f_{l_1}\cdots f_{l_{n+1}}} \text{ is given by}$$

$$\frac{u}{f_{i_1}\ldots f_{i_n}} \mapsto \sum_{\{i_1,\ldots,i_n\}=\{l_1,\ldots,\widehat{l_\alpha},\ldots l_{n+1}\}} (-1)^{\alpha-1}\frac{u f_{l_\alpha}}{f_{l_1}\ldots f_{l_{n+1}}}.$$

One immediate consequence of Theorem II.2.7 (c) is that local cohomology depends only on the radical of the associated ideal. Since the complex described in

(c) has only μ (possibly) nonzero terms, $H^i_I(M) = 0$ for indices i greater than the number of generators of any ideal with the same radical as I (and in particular, for all i greater than the minimal number of generators of I), which we call *the minimal number of generators of I, up to radical.*

Recall that for M an arbitrary (not necessarily finitely-generated) module over a ring R, we may define the *dimension* of M to be any of the following equal suprema:

$$\begin{aligned}
&\sup\{\dim(N) \mid N \subseteq M \text{ finitely-generated } R\text{-submodule}\}\\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}_R(M)\}\\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\}.
\end{aligned}$$

Besides the number of generators of I, up to radical, other familiar and important invariants of R, I, and M also restrict the vanishing of the local cohomology modules $H^i_I(M)$, as illustrated by the following theorem:

Theorem II.2.8. *Suppose that R is a Noetherian ring, I is an ideal of R, and that M is an R-module. Then*

(a) $H^i_I(M) = H^i_{\sqrt{I}}(M)$,

(b) $H^i_I(M) = 0$ *for i greater than the minimal number of generators of I, up to radical,*

(c) *If M is finitely generated and $M \neq IM$, then* $\min\{i \mid H^i_I(M) \neq 0\} = \operatorname{depth}_I(M)$,

(d) $H^i_I(M) = 0$ *for* $i > \dim M$, *and*

(e) *If $i > \dim R/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M)$, then $H^i_I(M) = 0$.*

Theorem II.2.9 (Long exact sequence in local cohomology)**.** *Suppose that R is a Noetherian ring, I is an ideal of R, and that $0 \to M' \to M \to M'' \to 0$ is an* *exact sequence of R-modules. Then there is a functorial long exact sequence in local cohomology:*

$$
\begin{tikzcd}
0 \arrow[r] & H_I^0(M') \arrow[r] & H_I^0(M) \arrow[r] & H_I^0(M'') \arrow[dll, rounded corners] \\
& H_I^1(M') \arrow[r] & H_I^1(M) \arrow[r] & H_I^1(M'') \arrow[dll, rounded corners] \\
& H_I^2(M') \arrow[r] & H_I^2(M) \arrow[r] & H_I^2(M'') \arrow[r] & \ldots
\end{tikzcd}
$$

$$
\ldots \longrightarrow H_I^i(M') \longrightarrow H_I^i(M) \longrightarrow H_I^i(M'') \longrightarrow \ldots
$$

Definition II.2.10 (Cohen-Macaulay ring)**.** A local ring $(R, \mathfrak{m})$ is *Cohen-Macaulay* if $\dim R = \operatorname{depth}_{\mathfrak{m}} R$. An arbitrary ring is *Cohen-Macaulay* if its localization at every prime ideal of R is a Cohen-Macaulay local ring.

Note that Theorem II.2.8 (c) and (d) combine to give a characterization of the Cohen-Macaulay property for a local ring in terms of local cohomology:

Corollary II.2.11. *A local ring $(R, \mathfrak{m})$ is Cohen-Macaulay if and only if the only nonzero local cohomology module $H_{\mathfrak{m}}^i(R)$ has index $i = \dim R$.*

Remark II.2.12. Note that Corollary II.2.11 implies that an arbitrary ring R is Cohen-Macaulay if and only if, for every prime ideal $\mathfrak{p}$ of R, the only nonzero local cohomology module of the form $H_{\mathfrak{p}R_{\mathfrak{p}}}^i(R_{\mathfrak{p}})$ has index $i = \dim R_{\mathfrak{p}}$.

Definition II.2.13 (Essential extension)**.** For R a ring, a homomorphism of R-modules $\phi : M \to N$ is an *essential extension* if it is both an injection and if every nonzero submodule of N has nonzero intersection with $\phi(M)$.

Lemma II.2.14. *Suppose that R is a ring and that M is an R-module. A maximal essential extension (an extension with no proper essential extension) $M \to E$ exists, and E is an injective R-module. If $M \xrightarrow{\phi} E$ and $M \xrightarrow{\phi'} E'$ are maximal essential* *extensions, then $E \cong E'$ as R-modules, and if ψ denotes such an isomorphism, the following diagram commutes:*

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[dr, "\phi'"'] & E \arrow[d, "\psi"] \\ & E' \end{tikzcd}$$

Lemma II.2.14 confirms that the following definition is well defined:

Definition II.2.15 (Injective hull)**.** If M is an R-module and $M \to E$ is a maximal essential extension, then E is called the *injective hull of M over R*, and is denoted $E_R(M)$.

Proposition II.2.16. *For $\mathfrak{p}$ a prime ideal of a ring R, $E_R(R/\mathfrak{p}) \cong E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$.*

Definition II.2.17 (Socle)**.** Let $(R, \mathfrak{m})$ be a local ring or let R be an $\mathbb{N}$-graded ring with $R_0 = k$, a field, and homogeneous maximal ideal $\mathfrak{m}$. Let M be an R-module. The *Socle of M*, denoted $\operatorname{Soc} M$, is the R-submodule $\operatorname{Ann}_M \mathfrak{m}$.

Note that $\operatorname{Soc} M$ is naturally a $R/\mathfrak{m}$-vector space.

Proposition II.2.18. *If R is a Noetherian ring and $\mathfrak{p}$ is a prime ideal of R, then $\operatorname{Ann}_{E_R(R/\mathfrak{p})} \mathfrak{p} \cong \operatorname{Frac}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p})$. In particular, if $(R, \mathfrak{m})$ is a local ring or R is an $\mathbb{N}$-graded ring with $R_0 = k$, a field, and homogeneous maximal ideal $\mathfrak{m}$, $\operatorname{Soc} E_R(k) \cong k$.*

Example II.2.19. If $R = k[x_1, \dots x_d]$ is the polynomial ring over a field k and $\mathfrak{m}$ is its homogeneous maximal ideal, then the complex described in Theorem II.2.7 (c) to define the local cohomology modules $H^i_\mathfrak{m}(R)$ is:

$$0 \to R \to \bigoplus_{i=1}^{n} R_{x_i} \to \bigoplus_{1 \le i < j \le n} R_{x_i x_j} \to \dots \to \bigoplus_{i=1}^{n} R_{x_1 \dots \widehat{x_1} \dots x_n} \to R_{x_1 \dots x_n} \to 0.$$

Since R is Cohen-Macaulay of dimension d, the only nonzero local cohomology module of R in this case is $H^d_\mathfrak{m}(R)$. Taking cohomology, $H^d_\mathfrak{m}(R) \cong R_{x_1 \dots x_n} / \sum_{i=1}^n R_{x_1 \dots \widehat{x_1} \dots x_n}$. A vector space basis for $H^d_\mathfrak{m}(R)$ consists of $x_1^{-\alpha_1} \cdot \dots \cdot x_n^{-\alpha_n}$, where each $\alpha_i > 0$;

this module has been called an "upside-down polynomial ring in $x_1, \dots, x_n$." In fact, $H^d_{\mathfrak{m}}(R) \cong E_R(k)$.

Definition II.2.20 (Gorenstein ring)**.** A *Gorenstein* local ring is a local ring with finite injective dimension as a module over itself. A Noetherian ring is *Gorenstein* if its localization at every prime ideal of the ring is a Gorenstein local ring.

The polynomial ring $k[x_1, \dots, x_n]$ from Example II.2.19 is Gorenstein. In fact, all regular rings, and so in particular, complete intersection rings, are Gorenstein. Gorenstein rings are Cohen-Macaulay. The fact that $H^d_{\mathfrak{m}}(R) \cong E_R(k)$, in fact, holds for *any* local or graded Gorenstein ring:

Theorem II.2.21. *Suppose that $(R, \mathfrak{m}, k)$ is a Gorenstein local ring, or that R is an $\mathbb{N}$-graded ring with $R_0 = k$, a field, and homogeneous maximal ideal $\mathfrak{m}$. If $\dim R = d$, then $H^d_{\mathfrak{m}}(R) \cong E_R(k)$.*

The following theorem is used in the proof of the Main Theorem on Minors V.10:

Theorem II.2.22. *If I is an ideal of a Noetherian ring R, M is an R-module, and S is a flat Noetherian R-algebra, then for any i, $H^i_I(M \otimes_R S) \cong H^i_I(M) \otimes_R S$.*

Corollary II.2.23. *If I is an ideal of a Noetherian ring R, M is an R-module, and $\Sigma \subseteq R$ is a multiplicative system, then for any i, $H^i_I(\Sigma^{-1}M) \cong \Sigma^{-1}H^i_I(M)$.*

Definition II.2.24 (Canonical module)**.** If $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring of dimension d, then a finitely-generated R-module ω is a *canonical module* for R if $\operatorname{Hom}_R(\omega, E_R(k)) \cong H^d_{\mathfrak{m}}(R)$ as R-modules. A finitely-generated R-module ω is a *canonical module* for an arbitrary Cohen-Macaulay ring R if, for every prime ideal $\mathfrak{p}$ of R, $\omega_{\mathfrak{p}}$ is a canonical module for $R_{\mathfrak{p}}$.

Theorem II.2.25 (Local duality)**.** *Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring of dimension d and let $E = H^d_{\mathfrak{m}}(R)$, which, by Theorem II.2.21, is an injective hull for k over*

R. Then for any finitely-generated R-module M, $H^i_{\mathfrak{m}}(M) \cong \operatorname{Hom}_R(\operatorname{Ext}^{d-i}_R(M,R), E)$ as functors in M.

Corollary II.2.26. *A Gorenstein local ring is its own canonical module.*

Proof. Apply Theorem II.2.25 to the case that M is the ring and i is its dimension. □

Theorem II.2.27. *Given a Cohen-Macaulay ring R, a canonical module for R exists if and only if R is a homomorphic image of a Gorenstein ring.*

Theorem II.2.28. *Assume that R is a Cohen-Macaulay domain that is a homomorphic image of a Gorenstein ring. Then the canonical module for R is isomorphic with an ideal of R containing a nonzerodivisor, so is a torsion-free module of rank one.*

II.3 Determinantal ideals

In this section, we briefly introduce a certain class of determinantal ideals.

Definition II.3.1 (The ideals $I_{t+1}(X)$)**.** Given a ring R and an $r \times s$ matrix $X = (r_{ij})$ of entries of R, then for $1 \leq t \leq r$, let $I_t(X)$ denote the ideal of R generated by the $t \times t$ minors of X.

Definition II.3.2 (Polynomial ring over a matrix)**.** If A is a ring and $X = [x_{ij}]$ is an $r \times s$ matrix of indeterminates, then let $A[X]$ denote the polynomial ring over A in the rs indeterminates x_{ij}.

The following theorem of Hochster and Eagon [HE71, Theorem 1] is used in the proof of the Main Theorem on Minors V.10.

Theorem II.3.3 (Hochster and Eagon)**.** *Assume that X is an $r \times s$ matrix of indeterminates, where $r \leq s$, that k is a field, and that $R = k[X]$. Then the ideals $I_t(X)$, $0 < t \leq r$, are prime. Additionally, $\operatorname{ht}_{I_t(X)} R = \operatorname{depth}_{I_t(X)} R = (r-t+1)(s-t+1)$.*

II.4 G-modules and $R[G]$-modules

This section reviews the relevant theory of G- and $R[G]$-modules; our main reference here is [Bor91].

Definition II.4.1 (Linear algebraic group)**.** A *linear algebraic group* over a field k is a Zariski-closed subgroup of $GL_n(k)$, for some positive integer n.

Definition II.4.2 (G-module, G-module action, G-submodule, simple G-module, G-module homomorphism, G-equivariant map)**.** Given a linear algebraic group G over a field k, a *G-module* is a k-vector space on which there exists a k-linear representation of G, a group homomorphism $\Phi : G \to GL(V)$. The corresponding group action $G \times V \to V$ on a G-module is called its *G-module action*. A *G-submodule* W of V is a k-vector subspace of V that is stable under its G-module action. A *simple G-module* is a nonzero G-module that contains no proper nonzero G-submodules. Given G-modules V and W, a *G-module homomorphism* $\phi : V \to W$ is a vector space homomorphism that is also *G-equivariant*, which means that for all $g \in G$ and $v \in V$, $g \cdot \phi(v) = \phi(g \cdot v)$.

Definition II.4.3 (Rational G-module)**.** Given a linear algebraic group G over a field k, a finite-dimensional G-module V is called a *rational G-module* if the action $G \times V \to V$ is a regular map of affine varieties over k. An arbitrary (possibly infinite-dimensional) G-module is a *rational G-module* if it is a directed union of G-stable finite-dimensional k-vector subspaces that are themselves rational G-modules.

A G-stable subspace of a rational G-module, a quotient of a rational G-module, or a direct sum of rational G-modules is again a rational G-module. If V and W are rational G-modules, then $V \otimes_k W$ is a rational G-module with action defined on simple tensors by $g \cdot (v \otimes w) = g \cdot v \otimes g \cdot w$. If V is also a finite-dimensional vector space, then $\mathrm{Hom}_k(V, W)$ is a rational G-module by $g \cdot f = gfg^{-1}$. Moreover, their definition implies that rational G-modules are also closed under directed unions.

Example II.4.4. For example, if $k = \mathbb{C}$, $G = GL_1(\mathbb{C}) = \mathbb{C}^\times$ and G acts on $V = \mathbb{C}$ by, for $\lambda \in G$ and $z \in V$, $\lambda \cdot z = \lambda^n z$ for some integer n, then this action makes V a rational G-module. However, the action defined by $\lambda \cdot z = \overline{\lambda} z$, where $\overline{\lambda}$ denotes the complex conjugate of λ, is *not* a rational G-module action over $\mathbb{C}$.

Remark II.4.5. Every linear algebraic group G acts rationally on the coordinate ring $k[G]$, and every finite-dimensional rational G-module occurs as a G-submodule of $k[G]^{\oplus h}$ for some h. Every finite-dimensional simple rational G-module occurs as a G-submodule of $k[G]$ [Fog69, Discussion following Definition 2.23].

Definition II.4.6 (Linearly reductive group)**.** A linear algebraic group G is called *linearly reductive* if every finite-dimensional rational G-module splits into a direct sum of simple G-modules.

In particular, if G is linearly reductive, every map of rational G-modules splits. Some examples of linearly reductive groups in characteristic zero are the general linear group (and, in particular, the multiplicative group of the field), the special linear group, the orthogonal group, the symplectic group, finite groups, and products of any of these. In characteristic $p > 0$, there are fewer linearly reductive groups; some examples are the multiplicative group of the field, finite groups whose orders are not multiples of p, and products of these.

Definition II.4.7 (W-isotypical component)**.** Given a linearly reductive group G, a rational G-module V, and a simple rational G-module W, the *W-isotypical component* of V is the direct sum of all G-submodules of V isomorphic to W, i.e., it is of the form $\bigoplus_i W_i \subseteq V$, where each $W_i \cong W$ as G-modules. As G-modules, V is the direct sum of its isotypical components.

Definition II.4.8 (Invariant part)**.** If V is a G-module, then the *invariant part of V*, denoted V^G, is the G-submodule of elements in V fixed by the action of G.

When G is linearly reductive, V^G is the isotypical component of k with the trivial action, so the functor on rational G-modules sending V to V^G is exact. The sum of all other isotypical components (the sum of all the simple G-submodules of V on which G does not act trivially) defines a unique G-module complement of V^G.

Definition II.4.9 ($R[G]$-module)**.** Let G be a linear algebraic group over a field k and let R be k-algebra that is a G-module. An R-module M that is also a G-module is an *$R[G]$-module* if for every $g \in G$, $r \in R$, and $u \in M$,

$$g(ru) = (gr)(gu).$$

Definition II.4.10 (Rational $R[G]$-module)**.** Given a field k and a k-algebra R with an action of a linear algebraic group G, a *rational $R[G]$-module* is an $R[G]$-module that is also a rational G-module.

Remark II.4.11. Every simple rational G-module occurs in the action of $G = SL_r(k)$ on $k[G] = k[x_{ij}]_{r\times r}/(\det([x_{ij}]_{r\times r}) - 1)$, and hence in the action on $k[x_{ij}]_{r\times r}$, which maps onto $k[G]$. Thus, all occur in the action on $k[X]$, where $X = [x_{ij}]_{r\times s}$ and $r \leq s$, which contains $k[x_{ij}]_{r\times r}$.

The following isomorphism of G-modules will be used in the proofs of Lemma II.4.13 and of Lemma III.8.

Remark II.4.12. Given a linear algebraic group G over a field k, and G-modules U and V, $\dim_k V < \infty$, we have the following isomorphism of G-modules:

$$\text{(II.4.12.1)} \qquad \begin{gathered} U \otimes_k V^* \cong \mathrm{Hom}_k(V, U) \\ u \otimes f \mapsto \phi, \text{ where } \phi(v) = f(v)u, \end{gathered}$$

where, given $g \in G$, for $u \otimes f \in U \otimes_k V^*$, $g \cdot (u \otimes f) = gu \otimes gf$, and for $\phi \in \mathrm{Hom}_k(V, U)$ and $v \in V$, $(g \cdot \phi)(v) = (g\phi g^{-1})(v)$.

Lemma II.4.13. *If G is a linearly reductive group over a field k, and U and V are rational G-modules, then*

$$(U \otimes_k V)^G \neq 0 \iff \textit{for some simple } G\textit{-submodule } W \textit{ of } V,\ W^* \hookrightarrow U.$$

Proof. Suppose that $(U \otimes_k V)^G \neq 0$ and that $U = \bigoplus_{i \in I} Y_i$ and $V = \bigoplus_{j \in J} W_j$ are decompositions into simple G-modules. Then $U \otimes_k V \cong \bigoplus_{i \in I, j \in J} Y_i \otimes_k W_j$ as G-modules, so that as G-modules,

$$(U \otimes_k V)^G \cong \bigoplus_{i \in I, j \in J} (Y_i \otimes_k W_j)^G.$$

For simple G-modules Y and W, $(Y \otimes_k W)^G = 0$ unless $W \cong Y^*$ as G-modules: By the adjointness of tensor and Hom, $(Y \otimes_k W)^* \cong \operatorname{Hom}_k(W, Y^*)$. Since G is linearly reductive, as G-modules,

$$\begin{aligned} \left((Y \otimes_k W)^G\right)^* &\cong ((Y \otimes_k W)^*)^G \\ &\cong (\operatorname{Hom}_k(W, Y^*))^G \\ &\cong \operatorname{Hom}_G(W, Y^*), \end{aligned}$$

the ring of G-module maps between W and Y^*. Thus, if $(Y \otimes_k W)^G \neq 0$, then $\operatorname{Hom}_G(W, Y^*) \neq 0$, so there exists a nonzero G-module map $W \to Y^*$, and since Y and W (and so also Y^*) are simple G-modules, the map must be an isomorphism. Thus, in our setup, $(U \otimes_k V)^G \neq 0$ implies $(Y_i \otimes_k W_j)^G \neq 0$ for some i and j, and so $Y_i \cong {W_j}^*$.

Now say that for some simple G-submodule W of V, $W^* \hookrightarrow U$. If we apply $(-) \otimes_k W$ to $W^* \hookrightarrow U$, we see that $W \otimes_k W^* \hookrightarrow W \otimes_k U$, and if we apply $(-) \otimes_k U$ to $W \subseteq V$, we have the injection $W \otimes_k U \hookrightarrow U \otimes_k V$. Thus, $W \otimes_k W^*$ injects into $U \otimes_k V$, so $(W \otimes_k W^*)^G$ injects into $(U \otimes_k V)^G$. Under (II.4.12.1), $W \otimes_k W^* \cong \operatorname{Hom}_k(W, W)$, and G acts by conjugation on $\operatorname{Hom}_k(W, W)$. The element of $W \otimes_k W^*$ that corresponds to the identity element of $\mathrm{Hom}_k(W, W)$ must be invariant, since $g\,\mathrm{id}_W\,g^{-1} = \mathrm{id}_W$, so $(W \otimes_k W^*)^G$ is nonzero, and $(U \otimes_k V)^G$ must also be nonzero.

$\square$

Corollary II.4.14. *Let G be a linearly reductive group. If V is a simple rational G-module and a rational G-module U has V-isotypical component $\widehat{U}$, then*

$$(U \otimes_k V^*)^G = \left(\widehat{U} \otimes_k V^*\right)^G.$$

Proof. Suppose that $U = \bigoplus_{i \in I} U_i$ as G-modules, where each U_i is a simple G-module. Let $J \subseteq I$ be the set of indices j such that $U_j \cong V$ as G-modules, so that $\widehat{U} = \bigoplus_{i \in J} U_i$. For $i \in I - J$, by Lemma II.4.13, $(U_i \otimes_k V^*)^G = 0$. This means that

$$(U \otimes_k V^*)^G = \left(\left(\bigoplus_{i \in J} U_i\right) \otimes_k V^*\right)^G = \left(\widehat{U} \otimes_k V^*\right)^G.$$

$\square$

II.5 Spectral sequence of a double complex

This section gives a brief introduction to the theory of a spectral sequence of a double complex; our main reference is [Wei94]. One specific such spectral sequence will be used in the proof of Theorem VI.3.

Without loss of generality, we will describe the theory when the complex is cohomological; the homological theory is analogous, with differentials lowering, rather than raising, degree. Suppose that $A^{\bullet\bullet}$ is a double complex such that for i or j negative, $A^{i,j} = 0$. Let $\partial^{i,j} : A^{i,j} \to A^{i,j+1}$ be the horizontal differential, and let $\delta^{i,j} : A^{i,j} \to A^{i+1,j}$ be the vertical differential.

We define a filtration on $A^{\bullet\bullet}$,

$$A^{\bullet\bullet} = \langle A^{\bullet\bullet}\rangle_0 \supseteq \langle A^{\bullet\bullet}\rangle_1 \supseteq \langle A^{\bullet\bullet}\rangle_2 \supseteq \dots \langle A^{\bullet\bullet}\rangle_p \supseteq \dots,$$

where $\langle A^{\bullet\bullet}\rangle_p$ is the same complex as $A^{\bullet\bullet}$, except with all positions in rows $i \leq p$ replaced by zeros. Maps are made zero as appropriate.

Recall that the total complex of $A^{\bullet\bullet}$ is the (single) complex $\mathcal{T}^{\bullet}(A^{\bullet\bullet})$ defined by

$$\mathcal{T}^n(A^{\bullet\bullet}) = \bigoplus_{i+j=n} A^{i,j},$$

and the differential $\mathcal{T}^n(A^{\bullet\bullet}) \to \mathcal{T}^{n+1}(A^{\bullet\bullet})$ is $\sum_{i+j=n} \partial^{i,j} + (-1)^i \delta^{i,j}$. The total complex $\mathcal{T}^{\bullet}(\langle A^{\bullet\bullet}\rangle_p)$ of each $\langle A^{\bullet\bullet}\rangle_p$ is a subcomplex of $\mathcal{T}^{\bullet}(A^{\bullet\bullet})$, so that the $\mathcal{T}^{\bullet}(\langle A^{\bullet\bullet}\rangle)_p :=$ $\mathcal{T}^{\bullet}(\langle A^{\bullet\bullet}\rangle_p)$ define a filtration of the total complex $\mathcal{T}^{\bullet}(A^{\bullet\bullet})$.

We define the complex $E_0^{\bullet}$ as $E_0^n = \bigoplus_p \langle \mathcal{T}^n(A^{\bullet\bullet})\rangle_p / \langle \mathcal{T}^n(A^{\bullet\bullet})\rangle_{p+1}$ an associated graded complex that consists of the direct sum of the rows. With $E_0^{p,q} = A^{p,q}$, the differential $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ is the row differential $\partial^{p,q}$. Let $E_1^{p,q}$ denote the cohomology with respect to $d_0^{p,q}$, so $E_1^n = \bigoplus_{p+q=n} E_1^{p,q}$ is the cohomology with respect to $d_0^n = \sum_{p+q=n} d_0^{p,q}$. Now, $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ is induced by the vertical differential $\delta^{p,q}$, and $E_2^{p,q}$ is the cohomology with respect to $d_1^{p,q}$. Analogously, $E_2^n = \bigoplus_{p+q=n} E_2^{p,q}$ is the cohomology with respect to $d_1^n = \sum_{p+q=n} d_1^{p,q}$. We may continue this process, where the cohomology with respect to the induced differential $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is $E_{r+1}^{p,q}$, and so $E_{r+1}^n = \bigoplus_{p+q=n} E_{r+1}^{p,q}$ is the cohomology with respect to $d_r^n = \sum_{p+q=n} d_r^{p,q}$. (The maps d_r^n move up one degree since $(p+r)+(q-r+1) = p+q+1$.) For example, the map $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ is akin to a knight's move in chess, as shown in Figure II.5.0.1. The sequence of complexes $E_r^{\bullet}$ is the *spectral sequence of the double complex* $A^{\bullet\bullet}$ with respect to the filtration.

The spectral sequence eventually stabilizes at each spot, i.e., for every p and q, there is some $\rho > 0$ such that $E_\rho^{p,q} = E_{\rho+i}^{p,q}$ for any $i \in \mathbb{N}$. The stable value at this spot

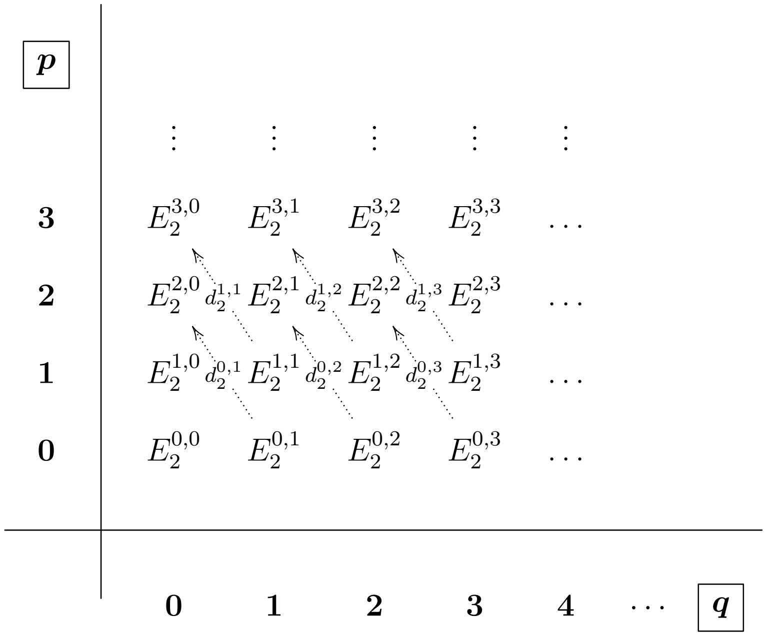

Figure II.5.0.1: The differentials $d_2^{p,q}$

is denoted $E_\infty^{p,q}$. Notationally, to denote this convergence, we write, for any fixed r_0,

$$E_{r_0}^{p,q} \underset{p}{\Longrightarrow} E_\infty^{p,q}.$$

Letting $E_\infty^n = \bigoplus_{p+q=n} E_\infty^{p,q}$, $E_\infty^\bullet$ is an associated graded complex of the cohomology of the total complex, $H^\bullet\left(\mathcal{T}^\bullet\left(A^{\bullet\bullet}\right)\right)$. Spectral sequences thus give us a way to translate information between each $E_{r_0}^\bullet$ term and this associated graded complex. Moreover, we could have also filtered by interchanging the roles of rows and columns, and the new $E_\infty^\bullet$ would again be an associated graded complex of $H^\bullet\left(\mathcal{T}^\bullet\left(A^{\bullet\bullet}\right)\right)$ with respect to a different grading, giving us a way to relate the intermediate terms of the two spectral sequences.

The following spectral sequence is utilized in the proof of Theorem VI.3: Suppose that I and J are ideals of a Noetherian ring R, and that M is an R-module. With respect to some choice of generators for I, let $\mathcal{C}(M; I)$ be the complex provided in Theorem II.2.7 (c) whose i^{th} cohomology is the local cohomology module $H_I^i(M)$, and define $\mathcal{C}(M; J)$ analogously. If we think of $\mathcal{C}(M; I)$ as a horizontal complex and $\mathcal{C}(M; J)$ as a vertical complex, the tensor product of the two will be a double complex $C^{\bullet\bullet}$. In the spectral sequence with respect to $C^{\bullet\bullet}$, each $E_2^{p,q}$ term is $H_J^p(H_I^q(M))$, and each $E_\infty^{p,q}$ is $H_{I+J}^{p+q}(M)$ [Har67, Proposition 1.4]:

Remark II.5.1 (Spectral sequence of iterated local cohomology modules)**.** Quite generally, if I and J are ideals of a Noetherian ring R, and M is an R-module, there is a spectral sequence

$$E_2^{p,q} = H_J^p(H_I^q(M)) \underset{p}{\Longrightarrow} E_\infty^{p,q} = H_{I+J}^{p+q}(M).$$

In particular, if $J = \mathfrak{m}$ is the homogeneous maximal ideal of a polynomial ring R over a field k, and $M = R$, only one $E_\infty^{p,q}$ will be nonzero (and will be isomorphic to $E_R(k)$), and will be at a position where $p + q = \dim R$. Studying the nonzero terms and the differentials of this spectral sequence in a specific case will allow us to relate its terms to gain information about certain iterated local cohomology modules of interest in the proof of Theorem VI.3.

CHAPTER III

Preliminaries

The "#" notation used in the following definition is not standard, but is very useful in our context.

Definition III.1 (Graded dual)**.** If k is a field and V is a $\mathbb{Z}$-graded k-vector space such that $\dim_k[V]_i < \infty$ for every $i \in \mathbb{Z}$, then the *graded dual of* V,

$$V^{\#} = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_k([V]_i, k),$$

is a $\mathbb{Z}$-graded k-vector space satisfying $\left[V^{\#}\right]_j = \operatorname{Hom}_k([V]_{-j}, k)$.

Note that if V is a finite-dimensional k-vector space, then $V^{\#} = V^* := \operatorname{Hom}_k(V, k)$. Note also that $(-)^{\#}$ is an exact contravariant functor.

Remark III.2. Suppose that k is a field and R is an $\mathbb{N}$-graded ring with $R_0 = k$ and homogeneous maximal ideal $\mathfrak{m}$. Suppose also that M is a $\mathbb{Z}$-graded Artinian R-module. Then $M_{\geq i} := \bigoplus_{n\geq i} M_n$ is a submodule of M, and since $\mathfrak{m}M_{\geq i} \subseteq M_{\geq i+1}$, each $M_i \cong M_{\geq i}/M_{\geq i+1}$ is a Noetherian R-module killed by $\mathfrak{m}$, so is a finite-dimensional k-vector space. Thus, M satisfies the hypotheses necessary to define its graded dual.

Remark III.3 (Graded dual is a rational $R[G]$-module)**.** Suppose that G is a linear algebraic group over a field k. Assume that R is an $\mathbb{N}$-graded ring such that $R_0 = k$, and that R is also a G-module, where G acts on R by k-automorphisms as to preserve its grading. Assume that M is a $\mathbb{Z}$-graded $R[G]$-module such that $\dim_k [M]_i < \infty$ for every i, and that the action of G preserves the grading of M. Then $M^{\#}$ is also a $\mathbb{Z}$-graded $R[G]$-module. For any $g \in G$, $f \in M^{\#}$, and $u \in M$,

$$(gf)(u) = f(g^{-1}u),$$

which is natural shorthand for $\sum_i f_i(g^{-1}u_i)$, assuming $f = \sum_i f_i$, where $\deg f_i = -i$, and $u = \sum_i u_i$, where $\deg u_i = i$. If M is a rational $R[G]$-module, this action of G makes $M^{\#}$ a rational $R[G]$-module as well.

Remark III.4. When k is a field and R is a Noetherian $\mathbb{N}$-graded ring with $R_0 = k$, $R^{\#} \cong E_R(k)$ as R-modules [BH93, Proposition 3.6.16].

Remark III.5. If k is a field and $R = k[x_1, \ldots, x_n]$ is a polynomial ring with homogeneous maximal ideal $\mathfrak{m}$, then $R^{\#} \cong E_R(k) \cong H^n_{\mathfrak{m}}(R)$ as R-modules (see Example II.2.19). However, if $H^n_{\mathfrak{m}}(R)$ is viewed as $R_{x_1 \cdots x_n} / \sum_{i=1}^n R_{x_1 \cdots \widehat{x_i} \cdots x_n}$, its grading is shifted: $H^n_{\mathfrak{m}}(R) \cong R^{\#}(-n)$ as $\mathbb{N}$-graded modules, where $\left[R^{\#}(-n)\right]_j = \left[R^{\#}\right]_{j-n}$.

Remark III.6 (Matlis Duality for graded modules)**.** Suppose that R is an $\mathbb{N}$-graded ring such that $R_0 = k$, and that M is a $\mathbb{Z}$-graded R-module. If M has DCC (respectively, ACC) as a graded module, then $M^{\#}$ has ACC (respectively, DCC). If M has either DCC or ACC, the natural map $M \to M^{\#\#}$ is an isomorphism of graded modules. Moreover, the functor $(-)^{\#}$ provides an anti-equivalence of categories from the category of $\mathbb{Z}$-graded R-modules with DCC to the category of $\mathbb{Z}$-graded R-modules with ACC, and vice versa [BH93, Theorem 3.6.17].

Lemma III.7. *Let G be a linearly reductive group over a field k. Let R be an $\mathbb{N}$-graded ring, with $R_0 = k$, that is also a G-module, where G acts on R by k-automorphisms as to preserve its grading. Let M be a $\mathbb{Z}$-graded $R[G]$-module such that the action* *of G respects the grading on M. If M has DCC or ACC, then the action of G on M and the induced action on $M^{\#\#}$ are compatible under the natural isomorphism $M \xrightarrow{\cong} M^{\#\#}$ given by Matlis duality (see Remark III.6).*

Proof. Under the map $M \xrightarrow{\cong} M^{\#\#} = \bigoplus_i \operatorname{Hom}_k(\operatorname{Hom}_k(M_i, k), k)$, $u = \sum_i u_i \in M$ maps to $\sum_i \phi_i$, where for any $f \in \operatorname{Hom}_k(M_i, k)$, $\phi_i(f) = f(u_i)$. For $g \in G$, we see that

$$((g\phi_i)(f))(u_i) = (\phi_i(g^{-1}f))(u_i) = f(gu_i).$$

□

Lemma III.8. *Suppose that U and V are $\mathbb{Z}$-graded rational G-modules, where $\dim_k V$ and each $\dim_k U_i$ are finite. Then as G-modules,*

$$(U \otimes_k V)^{\#} \cong U^{\#} \otimes_k V^*,$$

where $\left[(U \otimes_k V)^{\#}\right]_n$ precisely corresponds to $\left[U^{\#} \otimes_k V^*\right]_{-n}$ under the isomorphism.

Proof.

$$\begin{aligned}
(U \otimes_k V)^{\#} &\cong \bigoplus_n \operatorname{Hom}_k(\bigoplus_i V_i \otimes_k U_{n-i}, k) \\
&\cong \bigoplus_n \bigoplus_i \operatorname{Hom}_k(V_i \otimes_k U_{n-i}, k) \\
&\cong \bigoplus_n \bigoplus_i \operatorname{Hom}_k(V_i, \operatorname{Hom}_k(U_{n-i}, k)) \qquad \text{(III.8.1)} \\
&\cong \bigoplus_n \bigoplus_i V_i^* \otimes_k \operatorname{Hom}_k(U_{n-i}, k) \qquad \text{(III.8.2)} \\
&= \bigoplus_n \bigoplus_i [V^*]_{-i} \otimes_k \left[U^{\#}\right]_{-(n-i)} \\
&\cong \bigoplus_n \left[U^{\#} \otimes_k V^*\right]_{-n} \\
&\cong U^{\#} \otimes_k V^*.
\end{aligned}$$

(III.8.1) is the adjointness of the tensor and Hom functors, and (III.8.2) is the map (II.4.12). The only compatibility of the action that needs to be checked is (III.8.1); here, we check the "backward" map. Given $\psi : V_i \to \mathrm{Hom}_k(U_{n-i}, k)$,

$$g \cdot \psi : V_i \otimes_k U_{n-i} \to k \text{ is defined by}$$
$$v \otimes u \mapsto (g \cdot \psi)(v)(u),$$

where $(g \cdot \psi)(v)(u) = ((g\psi g^{-1})(v))(u) = (g\psi(g^{-1}v))(u) = \psi(g^{-1}v)(g^{-1}u)$. Under (III.8.1), ψ is sent to $\phi : V_i \otimes_k U_{n-k} \to k$, where, for $v \in V$, $\phi(v) = \psi(v)(u)$. This means that $g \cdot \phi(u \otimes v) = \phi(g^{-1}u \otimes g^{-1}v) = \psi(g^{-1}v)(g^{-1}u)$ as well.

□

CHAPTER IV

Proof of the Main Theorem

We prove the Main Theorem IV.8 in this section, which will be used (along with other tools) to prove the Main Theorem on Minors V.10 in Chapter V. Throughout this section, we need the following frequently-used hypothesis.

Hypothesis IV.1. *Let k be a field of characteristic zero, and let R be a $\mathbb{N}$-graded Noetherian ring such that $R_0 = k$, with homogeneous maximal ideal $\mathfrak{m}$. Let G be a linearly reductive group over k acting on R by k-automorphisms such that R is a rational G-module, and let M be a $\mathbb{Z}$-graded rational $R[G]$-module. Suppose that the actions of G on R and on M respect their gradings.*

Remark IV.2. Under Hypothesis IV.1, it may be easily verified that $\operatorname{Soc} M$ is a rational G-submodule of M, and so is also a rational $R[G]$-module.

We will shortly state and prove Key Lemma IV.4, a "rational $R[G]$-module version" of the following theorem of Lyubeznik:

Theorem IV.3 ([Lyu93, Theorem 3.4])**.** *Given a polynomial ring R over a field k of characteristic zero, an integer $n \geq 1$, and ideals $I_1, \ldots, I_n$ of R, an iterated local cohomology module*

$$M = H^{i_1}_{I_1}\left(H^{i_2}_{I_2}\left(\cdots\left(H^{i_n}_{I_n}(R)\right)\cdots\right)\right)$$

has only finitely many associated primes contained in a given maximal ideal of R. If M is supported only at the homogeneous maximal ideal $\mathfrak{m}$, then M is isomorphic to a finite direct sum of copies of $E_R(k)$.

In particular, this holds when M is any local cohomology module $H^i_I(R)$ that is supported only at $\mathfrak{m}$, or when M is any $H^0_{\mathfrak{m}}(H^i_I(R))$.

Key Lemma IV.4. *Suppose that $R, \mathfrak{m}, G$ and M satisfy Hypotheses IV.1, and that M is also an injective Artinian R-module supported only at $\mathfrak{m}$. Let $V = \operatorname{Soc} M$. Then there exists a G-submodule $\widetilde{V^{\#}}$ of $M^{\#}$ (see Definition III.1) such that $\widetilde{V^{\#}} \cong V^{\#}$ as rational G-modules, and as rational $R[G]$-modules,*

$$M \cong \left(R \otimes_k \widetilde{V^{\#}}\right)^{\#},$$

where $\left(R \otimes_k \widetilde{V^{\#}}\right)^{\#} \cong R^{\#} \otimes_k V$ as rational G-modules.

Proof. If $x_1, \ldots, x_n$ generate $\mathfrak{m}$, we have an exact sequence of R-modules:

$$0 \longrightarrow V \xrightarrow{i} M \xrightarrow{\theta} M^{\oplus n},$$

where i is the inclusion of rational $R[G]$-modules, and $\theta(u) = (x_1 u, \ldots, x_n u)$ for $u \in M$. By taking graded duals, we obtain the following exact sequence of R-modules:

$$(M^{\oplus n})^{\#} \xrightarrow{\theta^{\#}} M^{\#} \xrightarrow{i^{\#}} V^{\#} \longrightarrow 0,$$

where $i^{\#}$ is also a map of rational $R[G]$-modules. Under the canonical isomorphism $(M^{\oplus n})^{\#} \cong \left(M^{\#}\right)^{\oplus n}$, given $f_1, \ldots, f_n \in M^{\#}$,

$$\theta^{\#}(f_1, \ldots, f_n) = x_1 f_1 + \ldots + x_n f_n.$$

This means that $\mathrm{Im}(\theta^{\#}) = \mathfrak{m}M^{\#}$, and

$$0 \longrightarrow \mathfrak{m}M^{\#} \longrightarrow M^{\#} \xrightarrow{i^{\#}} V^{\#} \longrightarrow 0$$

is an exact sequence of rational $R[G]$-modules, so $V^{\#} \cong M^{\#}/\mathfrak{m}M^{\#}$ as rational $R[G]$-modules. Moreover, since G is linearly reductive, the map $i^{\#}$ has a splitting, ϕ, as a map of G-modules:

$$\begin{tikzcd} M^{\#} \arrow[r, shift left, "i^{\#}"] & V^{\#} \arrow[l, shift left, "\phi"] \end{tikzcd}.$$

If $\widetilde{V^{\#}} \subseteq M^{\#}$ is the image of ϕ, then $\widetilde{V^{\#}} \cong V^{\#} \cong M^{\#}/\mathfrak{m}M^{\#}$ as rational G-modules.

Since M is Artinian, $M^{\#}$ is Noetherian, and by Nakayama's lemma, a k-basis for $\widetilde{V^{\#}}$ generates $M^{\#}$ minimally as an R-module. Since $R \otimes_k \widetilde{V^{\#}}$ is an R-module via, for $s \in R$ and $\sum_i r_i \otimes v_i \in R \otimes_k \widetilde{V^{\#}}$, $s \cdot \sum_i r_i \otimes v_i = \sum_i sr_i \otimes v_i$, as R-modules,

$$R \otimes_k \widetilde{V^{\#}} \twoheadrightarrow M^{\#}, \text{ where} \tag{IV.4.1}$$
$$\sum_i r_i \otimes v_i \mapsto \sum_i r_i v_i.$$

For $\sum_i r_i \otimes v_i \in R \otimes_k \widetilde{V^{\#}}$, $g \cdot \sum_i r_i \otimes v_i = \sum_i g \cdot r_i \otimes g \cdot v_i$, and since $M^{\#}$ is an $R[G]$-module, $g \cdot \sum_i r_i v_i = \sum_i (g \cdot r_i)(g \cdot v_i)$ in $M^{\#}$, so (IV.4.1) is a surjection of rational G-modules.

By Theorem IV.3, $M \cong E_R(k)^{\oplus \alpha}$ for some $\alpha \in \mathbb{N}$. As $\mathrm{Ann}_{E_R(k)} \mathfrak{m} \cong k$ (see Proposition II.2.18), $V = \mathrm{Soc}\, M$ is a finite-dimensional k-vector space of dimension α. This means that $\widetilde{V^{\#}}$ (which is isomorphic to V^* as G-modules) also has dimension α over k, and since $M^{\#} \cong R^{\oplus \alpha}$ (see Remark III.4), (IV.4.1) must be an isomorphism. Since M has DCC, noting that the G-module structures of M and $M^{\#\#}$ are compatible by Lemma III.7, by taking graded duals, we have that $M \cong M^{\#\#} \cong \left(R \otimes_k \widetilde{V^{\#}}\right)^{\#}$ as rational $R[G]$-modules. Moreover, by Lemma III.8, as rational G-modules,

$$\left(R \otimes_k \widetilde{V^{\#}}\right)^{\#} \cong R^{\#} \otimes_k \left(\widetilde{V^{\#}}\right)^{*} \cong R^{\#} \otimes_k V^{**} \cong R^{\#} \otimes_k V.$$

□

Lemma IV.5. *Suppose that G is a linearly reductive group and that R is a k-vector space that is a $\mathbb{Z}$-graded G-module, such that G preserves its grading and $\dim_k R_i < \infty$ for all $i \in \mathbb{Z}$. Suppose that V is a G-module. If some simple G-submodule of V is a G-submodule of R, then*

$$(R^\# \otimes_k V)^G = 0 \iff R^\# \otimes_k V = 0.$$

In particular, if R, $\mathfrak{m}$, G, and M satisfy Hypothesis IV.1, and M is also an injective Artinian R-module supported only at $\mathfrak{m}$, then

$$M^G = 0 \iff M = 0.$$

Proof. The backward implication clearly holds. For the forward implication, suppose that $R^\# \otimes_k V \neq 0$ and that a simple G-submodule W of V is also a G-submodule of R, so that $W \hookrightarrow R_n$, for some n, as G-modules. Dualizing, $R^\# \supseteq {R_n}^\# \twoheadrightarrow W^*$, which splits as G-modules since G is linearly reductive, so $W^* \hookrightarrow R^\#$. Thus, by Lemma II.4.13, $(R^\# \otimes_k V)^G \neq 0$.

The last statement can be seen by applying the result to the case when $V = \operatorname{Soc} M$ and noting Lemma IV.4.

□

Lemma IV.6. *Suppose that R, $\mathfrak{m}$, G, and M satisfy Hypothesis IV.1 and that M is also a nonzero injective Artinian R-module supported only at $\mathfrak{m}$. Assume that all simple G-submodules of $\operatorname{Soc} M$ are also G-submodules of R. If*

$$\operatorname{Soc} M = V_1 \oplus \ldots \oplus V_\alpha$$

as G-modules, where each V_i is nonzero, then

$$M = (R^{\#} \otimes_k V_1) \oplus \ldots \oplus (R^{\#} \otimes_k V_\alpha)$$

as rational $R[G]$-modules, where each $R^{\#} \otimes_k V_i$ is nonzero, and

$$M^G = (R^{\#} \otimes_k V_1)^G \oplus \ldots \oplus (R^{\#} \otimes_k V_\alpha)^G$$

as R^G-modules, where each $(R^{\#} \otimes_k V_i)^G$ is nonzero. In particular, if M^G is a indecomposable R^G-module, then $\operatorname{Soc} M$ is a simple G-module.

Proof. The first implication follows from Lemma IV.4 after applying $R^{\#} \otimes_k (-)$. The second follows by applying $(-)^G$ and noting that each summand is nonzero by Lemma IV.5. □

Lemma IV.7. *Let G be a linearly reductive group over a field k acting on a k-algebra R, and let N be a rational $R[G]$-module. Let $J = (f_1, \ldots, f_n)$ be an ideal of R^G, and let $I = JR$. Then for every index i, $H^i_I(N)$ is also a rational $R[G]$-module, and every simple G-submodule of $H^i_I(N)$ is also a G-submodule of N. Moreover, there is a canonical isomorphism of R^G-modules*

$$\left(H^i_I(N)\right)^G \cong H^i_J\left(N^G\right).$$

Proof. Since N is a G-module, for $f \in R^G$, $N_f = \varinjlim \left(N \xrightarrow{\cdot f} N \xrightarrow{\cdot f} N \xrightarrow{\cdot f} \ldots\right)$ as G-modules: If $g \in G$ and $\frac{u}{f^m} \in N_f$, which corresponds to $[u]$ in the mth copy of N in the direct limit, then $g \cdot \frac{u}{f^m} = \frac{g \cdot u}{f^m}$, which corresponds to $[g \cdot u]$ in the mth copy of N in the direct limit.

Hence, N_f is a G-module such that all simple G-submodules of N_f are also G-submodules of N. Since products of any of the f_j are fixed by G, every term in the following complex (see Theorem II.2.7) is a G-module:

$$0 \to N \xrightarrow{\delta_0} \bigoplus_{j=1}^{n} N_{f_j} \xrightarrow{\delta_1} \ldots \xrightarrow{\delta_{n-2}} \bigoplus_{j=1}^{n} N_{f_1 \ldots \widehat{f_j} \ldots f_n} \xrightarrow{\delta_{n-1}} N_{f_1 f_2 \ldots f_n} \to 0.$$

Since the maps δ_j on each summand are, up to a sign, further localization maps, they are G-equivariant. This makes the cohomology modules, $H_I^i(N)$, G-modules as well, and they inherit the property that all their simple G-submodules are also G-submodules of N. Additionally, these local cohomology modules are $R[G]$-modules since N is one: given any $g \in G$, $r \in R$, and $\left[\frac{u}{f^m}\right] \in H_I^i(N)$,

$$g\left(r\left[\frac{u}{f^m}\right]\right) = g\left[\frac{ru}{f^m}\right] = \left[\frac{(gr)(gu)}{f^m}\right] = (gr)\left(g\left[\frac{u}{f^m}\right]\right).$$

For the last statement, first notice that for any $f \in R^G$ (e.g., the product of any of the f_j), $\left(N^G\right)_f = (N_f)^G$. Taking invariants also commutes with taking direct sums, so $H_J^i\left(N^G\right)$ is isomorphic the cohomology of the complex

$$0 \to N^G \xrightarrow{d_0} \left(\bigoplus_{j} N_{f_j}\right)^G \xrightarrow{d_1} \ldots \xrightarrow{d_{n-2}} \left(\bigoplus_{j} N_{f_1 \ldots \widehat{f_j} \ldots f_n}\right)^G \xrightarrow{d_{n-1}} (N_{f_1 f_2 \ldots f_n})^G \to 0,$$

where d_i is the restriction of δ_i to the invariant part of the i^{th} module in the complex. Since G is linearly reductive, the functor $V \mapsto V^G$ of G-modules is exact, and we may conclude that $(H_I^i(N))^G \cong H_J^i\left(N^G\right)$. □

Main Theorem IV.8. *Let R be a polynomial ring over a field k of characteristic zero with homogeneous maximal ideal $\mathfrak{m}$. Let G be a linearly reductive group over k acting by degree-preserving k-automorphisms on R, such that R is a rational G-module. Assume that $A = R^G$ has homogeneous maximal ideal $\mathfrak{m}_A$, let $d = \dim A$, and let $I = \mathfrak{m}_A R$. Then $H_I^d(R) \neq 0$ and I is generated up to radicals by d elements and not fewer, so that $H_I^i(R) = 0$ for $i > d$. Moreover, the following hold:*

(a) *If $i < d$, then $\mathfrak{m}$ is not an associated prime of $H^i_I(R)$; i.e., $H^0_{\mathfrak{m}}(H^i_I(R)) = 0$.*

If $H^d_I(R)$ is supported only at $\mathfrak{m}$ (e.g., this holds if, after localization at any of the indeterminates of R, I requires fewer than d generators up to radical), then

(b) *$V := \operatorname{Soc} H^d_I(R)$ is a simple G-module, and*

(c) *As rational $R[G]$-modules, $H^d_I(R) \cong E_R(k) \otimes_k V$.*

Proof. By Lemma IV.7, we know that for every i, $(H^i_I(R))^G \cong H^i_{m_A}(A)$ as R^G-modules. The invariant part of $H^d_I(R)$, $H^d_{m_A}(A)$, is nonzero since $d = \dim A$, so $H^d_I(R) \neq 0$. The maximal ideal $\mathfrak{m}_A$ of A is generated, up to radical, by $d = \dim A$ elements, so its expansion to R, $I = \mathfrak{m}_A R$, will also be generated up to radical by the same d elements: If $\mathfrak{m}_A = \sqrt{(f_1, \ldots, f_d)}$ in A, then $\sqrt{I} = \sqrt{\mathfrak{m}_A R} = \sqrt{(f_1, \ldots, f_d)}$ in R, since if some $x \in \sqrt{(f_1, \ldots, f_d)} \subset R$, then x^N is in the ideal $(f_1, \ldots, f_d)$ of R for some N, which sits inside the ideal $(f_1, \ldots, f_d)A$ expanded to R, which is itself inside $\sqrt{(f_1, \ldots, f_d)}R = \mathfrak{m}_A R$.

For part (a), assume that $i < d$. By Lemma IV.7, $H^i_I(R)$ is a rational $R[G]$-module, so its submodule of elements killed by some power of $\mathfrak{m}$, $H^0_{\mathfrak{m}}(H^i_I(R))$, is also a rational $R[G]$-module. By definition, $(H^0_{\mathfrak{m}}(H^i_I(R)))^G$ is the R^G-submodule of $H^i_I(R)$ consisting of invariant elements that are killed by some power of $\mathfrak{m}$; thus, it is the R^G-submodule of $H^i_I(R)^G \cong H^i_{\mathfrak{m}_A}(A)$ (by Lemma IV.7) consisting of elements killed by a power of $\mathfrak{m}$. By the theorem of Hochster and J. Roberts [HR74, Main Theorem] or of Boutot [Bou87, Théorème], since G is linearly reductive and $A = R^G$, A must be Cohen-Macaulay. Since $i < d = \dim A$, $H^i_{\mathfrak{m}_A}(A) = 0$, so in particular, its submodule $(H^0_{\mathfrak{m}}(H^i_I(R)))^G$ must also vanish. By Theorem IV.3, $H^0_{\mathfrak{m}}(H^i_I(R))$ is isomorphic to a finite direct sum of copies of $E_R(k)$. Therefore, by Lemma IV.5, since its invariant part vanishes, $H^0_{\mathfrak{m}}(H^i_I(R))$ must also vanish.

Now suppose that $H^d_I(R)$ is supported only at $\mathfrak{m}$. Since R (and so A also) is a domain, the canonical module of A (which exists since A is a homomorphic image of a polynomial ring by Theorem II.2.27), ω_A, must be rank one and isomorphic to an ideal of A, so torsion-free (see Theorem II.2.28). Thus, ω_A is an indecomposable A-module. (If $\omega_A = M \oplus N$, since its rank is one, tensoring with $\operatorname{Frac}(A)$ must kill either M or N. However, elements of A cannot kill those of ω_A, as it is isomorphic to an ideal of a domain.) Therefore,

$$\operatorname{Hom}_A(\omega_A, E_A(A/\mathfrak{m}_A)) \cong H^d_{\mathfrak{m}_A}(A) \cong (H^d_I(R))^G$$

must also be indecomposable, so $\operatorname{Soc} H^d_I(R)$ is a simple G-module by Lemma IV.6.

Part (c) is a restatement of the second part of Key Lemma IV.4.

$\square$

CHAPTER V

Proof of the Main Theorem on Minors

Remark V.1. If a topological group G acts continuously on a topological space Z permuting a finite collection of closed sets $V_1, \ldots, V_m \subseteq Z$, then G must fix each V_i: For each $1 \leq i \leq m$ and $v \in V_i$, the map $\Theta_v : G \to Z$ given by $\Theta_v(g) = g \cdot v$ is continuous, so $\Theta_v^{-1}(V_i) = \{g \in G \mid g \cdot v \in V_i\}$ is closed in G. Similarly, if $\Theta^v : G \to \mathbb{A}^n$ is given by $\Theta^v(g) = g^{-1} \cdot v$, then the set $(\Theta^v)^{-1}(V_i)$ is also closed. Thus, the sets $\bigcap_{v \in V_1} \Theta_v^{-1}(V_i) = \{g \in G \mid gV_i \subseteq V_i\}$ and $\bigcap_{v \in V_1} (\Theta^v)^{-1}(V_i) = \{g \in G \mid V_i \subseteq g \cdot V_i\}$ are closed in G, so their intersection, $\{g \in V_i \mid g \cdot V_i = V_i\} = \operatorname{stab}_G V_i$ (the stabilizer of V_i in G) is also closed in G.

As G permutes the V_i, we have a map $\phi : G \to S_m$ (the symmetric group on m letters). Since $\phi^{-1}(\operatorname{stab}_G(V_i)) = \operatorname{stab}_{S_m}(i)$, ϕ induces $G/\operatorname{stab}_G(V_i) \hookrightarrow S_m/\operatorname{stab}_{S_m}(i)$. As $S_m/\operatorname{stab}_{S_m}(i)$ is finite, so is $G/\operatorname{stab}_G(V_i)$, and each $\operatorname{stab}_G(V_i)$ is a finite index subgroup of G. If $\operatorname{stab}_G(V_i) \subsetneq G$, then since G is closed, its cosets would disconnect G, which is impossible. Thus, each $\operatorname{stab}_G(V_i) = G$, and G fixes each V_i.

Lemma V.2. *Let G be a connected linear algebraic group, let R be a rational G-module, and let M be an $R[G]$-module such that $\operatorname{Ass}_R(M)$ is finite. Then every associated prime of M is stable under the action of G.*

Proof. Suppose that $\mathfrak{p} = \operatorname{Ann}_R u$, for some $u \in M$. We claim that for any $g \in G$, $g \cdot \mathfrak{p} = \operatorname{Ann}_R(gu)$. To see that $g \cdot \mathfrak{p} \subseteq Ann_R(gu)$, take $x \in \operatorname{Ann}_R u$. Since M is an $R[G]$-module, we have that $(gx)(gu) = g(xu) = g \cdot 0 = 0$. To see the opposite inclusion, suppose that $x \in \mathrm{Ann}_R(gu)$. Then $x = g(g^{-1}x)$, and $g^{-1}x \in \mathfrak{p} = \mathrm{Ann}_R(u)$:

$$\begin{aligned}(g^{-1}x)u &= (g^{-1}g)((g^{-1}x)u) \\ &= g^{-1}(g((g^{-1}x)u)) \\ &= g^{-1}(x(gu)),\end{aligned}$$

which is zero. Moreover, $g \cdot \mathfrak{p}$ is prime: Take $x, y \in R$ such that $xy \in g \cdot \mathfrak{p}$, so $xy = ga$, $a \in \mathfrak{p}$. Then $(g^{-1}x)(g^{-1}y) = g^{-1}(xy) = a \in \mathfrak{p}$, so either $g^{-1}x \in \mathfrak{p}$ or $g^{-1}y \in \mathfrak{p}$; i.e., either $x \in g \cdot \mathfrak{p}$ or $y \in g \cdot \mathfrak{p}$. Therefore, G acts on the set $\mathrm{Ass}_R(M)$.

For $g \in G$, the map $\phi_g : R \to R$, where, for $r \in R$, $\phi_g(r) = g \cdot r$, induces a natural map $\psi_g : \operatorname{Spec} R \to \operatorname{Spec} R$, where $\psi_g(\mathfrak{p}) = \phi_g^{-1}(\mathfrak{p})$, giving a continuous action of G on $\operatorname{Spec} R$. We know that G acts on the finite set $\mathrm{Ass}_R(M)$; suppose that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are all the associated primes of M in R. If, for some $g \in G$, $g \cdot \mathfrak{p}_i = \mathfrak{p}_j$, then under the action of G on $\operatorname{Spec} R$, $g \cdot \mathbb{V}(\mathfrak{p}_i) = \mathbb{V}(\mathfrak{p}_j)$, which means that G acts on the collection of $\mathbb{V}(\mathfrak{p}_1), \ldots, \mathbb{V}(\mathfrak{p}_m)$. Thus, by Remark V.1, G acts trivially on $\mathrm{Ass}_R(M)$. □

Hypothesis V.3. *Let k be a field of characteristic zero, let X be an $r \times s$ matrix of indeterminates, where $r < s$, and let $R = k[X]$ be the polynomial ring over k in the entries of X. For $0 < t \le r$, let $I_t(X)$ be the ideal of R generated by the $t \times t$ minors of X, which is prime by* [HE71, Theorem 1]*. Furthermore, let $I = I_r(X)$ be the ideal generated by the maximal minors of X.*

Remark V.4 (Square matrix case)**.** Suppose that R satisfies Hypothesis V.3, but assume instead that $r = s$. Here, $I = (\Delta)$, where Δ is the determinant of X, and the only nonzero local cohomology module is $H^1_I(R)$, which is isomorphic to R_Δ/R.

Remark V.5 (Action of the special linear group on the polynomial ring of interest)**.** Let k, R, and I satisfy Hypothesis V.3, and let $G = SL_r(k)$, which is linearly reductive since the characteristic of k is zero. Considering $\Gamma \in G$ as an $r \times r$ matrix, the action of Γ on the k-algebra R is defined by where the entries of X are sent. The $(\alpha, \beta)^{\text{th}}$ entry of X is sent to the $(\alpha, \beta)^{\text{th}}$ entry of $\Gamma \cdot X$. Thus, G acts by k-algebra automorphisms that correspond to invertible row operations on the matrix X. Additionally,

- The maximal minors of X are fixed by the action of G, so the ideal I generated by them is G-stable. Moreover, a classical invariant theory result of Weyl states that R^G is the k-subalgebra of R generated over k by the maximal minors of X [Wey39, Theorem 2.6.A]. This means that $I = \mathfrak{m}_{R^G} R$, where $\mathfrak{m}_{R^G}$ is the homogeneous maximal ideal of R^G.

- In fact, R^G is the homogeneous coordinate ring of the Plücker embedding of the Grassmann variety of r-planes in s-space, which has dimension $r(s-r)$; therefore, $\dim R^G = r(s-r)+1$.

Remark V.6. Under Hypothesis V.3, R is a rational $R[G]$-module: Since the action of G is induced by that on the linear forms, R is certainly an $R[G]$-module. Moreover, R will be the directed union of $V_n := \bigoplus_{i \le n} R_i$, each a finite-dimensional G-module.

Lemma V.7. *Suppose that k, R, and I satisfy Hypothesis V.3, and assume that $d = r(s-r)+1$. Then $\operatorname{Soc} H^d_I(R)$ is a one-dimensional k-vector space.*

Proof. Let $G = SL_r(k)$ act on R as in Remark V.5. Let $A = R^G$, let $\mathfrak{m}_A$ be its homogeneous maximal ideal, and let $V = \operatorname{Soc} H^d_I(R)$.

Since A is the ring of invariants of the semisimple group G, A is Gorenstein by the theorem of Hochster and J. Roberts [HR74, Corollary 1.9]. Therefore, its canonical module is isomorphic to A (see Corollary II.2.26), and as G-modules,

$$(R \otimes_k V^*)^G \cong \left((R \otimes_k V^*)^{\#\#}\right)^G \tag{V.7.1}$$

$$\cong \left((R^{\#} \otimes_k V)^{\#}\right)^G \tag{V.7.2}$$

$$\cong \left(\left(H_I^d(R)\right)^{\#}\right)^G \tag{V.7.3}$$

$$\cong \left(\left(H_I^d(R)\right)^G\right)^{\#} \tag{V.7.4}$$

$$\cong \left(H_{\mathfrak{m}_A}^d(A)\right)^{\#}, \tag{V.7.5}$$

which is isomorphic to A. (V.7.1) is by Matlis Duality (see Remark III.6). (V.7.2) and (V.7.3) are by Key Lemma IV.4, (V.7.4) holds since G is linearly reductive, and (V.7.5) is due to Lemma IV.7.

By Main Theorem IV.8 (b), $\operatorname{Soc} H_I^d(R)$ is a simple G-module. Let W be the V-isotypical component of R. Then W is graded, and is isomorphic to, as G-modules, a finite sum of copies of V in each degree. Say $W_j \cong \bigoplus_{n_j} V$ is the V-isotypical component of R_j. We have the following graded isomorphisms as G-modules:

$$\begin{aligned}
A &\cong (V^* \otimes_k R)^G \\
&\cong (V^* \otimes_k W)^G && \text{(V.7.6)} \\
&\cong \left(V^* \otimes_k \left(\bigoplus_j \bigoplus_{n_j} V\right)\right)^G \\
&\cong \bigoplus_d \bigoplus_{n_j} (V^* \otimes_k V)^G \\
&\cong \bigoplus_j \bigoplus_{n_j} (\operatorname{Hom}_k(V,V))^G && \text{(V.7.7)} \\
&\cong \bigoplus_j \bigoplus_{n_j} \operatorname{Hom}_G(V,V),
\end{aligned}$$

the G-module maps from V to itself. (V.7.6) is due to Lemma II.4.13 and (V.7.7) is as in Remark II.4.12.

Since A has a one-dimensional vector space in least degree (degree zero), we have that $\operatorname{Hom}_G(V,V) \cong k$. This corresponds to exactly one copy of V in least degree in the simple G-module decomposition of R (i.e., for μ the least degree m for which V

appears as a G-submodule of R_m, $W_\mu \cong V$, so $n_\mu = 1$).

In fact, if a nontrivial G-module occurs in R, it occurs with multiplicity greater than one in the smallest degree of R in which it occurs. (This is, for example, a consequence of [GW98, Theorem 5.2.7].) This contradiction implies that V must be a trivial G-module. Since V is also a simple G-module by Main Theorem IV.8 (b), it must be a one-dimensional k-vector space. □

Remark V.8 (Another useful group action on the polynomial ring of interest)**.** Let k and R satisfy Hypothesis V.3, and let H be the connected group $SL_r(k) \times SL_s(k)$ [GW98, Theorem 2.19]. Then H acts on R by k-algebra automorphisms as follows: Considering $\Gamma \in SL_r(k)$ and $\Gamma' \in SL_s(k)$ as $r \times r$ and $s \times s$ matrices, respectively, the action sends the entries of X to those of $\Gamma X \left(\Gamma'\right)^{-1}$.

- The action of $SL_r(k) \times \mathrm{Id}_{r\times r}$ sends $x_{\alpha\beta}$ to any linear combination of the indeterminates in its row, and that of $\mathrm{Id}_{s\times s} \times SL_s(k)$ does the same with respect to columns; therefore, under the action of H, any $x_{\alpha\beta}$ is sent to any other $x_{\alpha'\beta'}$.

The following observation is used in the proof of the Main Theorem on Minors V.10.

Remark V.9. Let k, R, I, and $I_t(X)$ satisfy Hypothesis V.3. Over $R_{x_{11}}$, we can perform elementary row and column operations on X (first subtract $\frac{x_{1\beta}}{x_{11}}$ times the first column from the βth column of the matrix, $\beta > 1$, and then subtract $\frac{x_{\alpha 1}}{x_{11}}$ times the first row from the αth row, $\alpha > 1$, and finally, scale the first row), to obtain the matrix

$$\left[\begin{array}{c|ccc} 1 & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & Y & \\ 0 & & & \end{array}\right],$$

where Y is the $(r-1) \times (s-1)$ matrix $\left[x_{\alpha\beta} - \frac{x_{1\beta}}{x_{11}} x_{\alpha 1}\right]_{\substack{1<\alpha\leq r \\ 1<\beta\leq s}}$.

Let $S = k\left[y_{\alpha\beta}, x_{11}, x_{11}^{-1}, x_{\alpha 1}, x_{1\beta} \mid 1 < \alpha \leq r, 1 < \beta \leq s\right]$. Since only elementary row and column operations were used to transform the matrix, they are invertible, and the transformation defines an isomorphism

$$S \to R_{x_{11}}, \text{ where} \tag{V.9.1}$$

$$y_{\alpha\beta} \mapsto x_{\alpha\beta} - \frac{x_{1\beta}}{x_{\alpha 1}} x_{11}.$$

Under (V.9.1), the ideal $I_{t+1}(X)$ of $R_{x_{11}}$ corresponds precisely to the ideal $I_t(Y)$ of S. Thus, (V.9.1) induces the isomorphism $\left(H^i_{I_r(X)}(R)\right)_{x_{11}} \cong H^i_{I_{r-1}(Y)}(S)$ for any i. Since $H^i_{I_{r-1}(Y)}(S) \cong H^i_{I_{r-1}(Y)}\left(k[Y] \otimes_{k[Y]} S\right)$, which is, in turn, isomorphic to $H^i_{I_{r-1}(Y)}(k[Y]) \otimes_{k[Y]} S$ since S is flat over $k[Y]$ (see Theorem II.2.22), we have that

$$\left(H^i_{I_r(X)}(R)\right)_{x_{11}} \cong H^i_{I_{r-1}(Y)}(k[Y]) \otimes_{k[Y]} S. \tag{V.9.2}$$

Main Theorem on Minors V.10. *Let k, R, I, and $I_t(X)$ satisfy Hypothesis V.3.*

(a) *Let $d = \max\{i : H^i_I(R) \neq 0\}$, so that $d = r(s-r)+1$ by Theorem I.1.2. Then*

$$H^d_I(R) \cong E_R(k).$$

(b) *$H^i_I(R) \neq 0$ if and only if $i = (r-t)(s-r)+1$ for some $0 \leq t < r$.*

(c) *Furthermore, if $i = (r-t)(s-r)+1$, then*

$$H^i_I(R) \hookrightarrow E_R(R/I_{t+1}(X)) \cong H^i_I(R)_{I_{t+1}(X)}.$$

In particular, $\mathrm{Ass}_R\left(H^i_I(R)\right) = \{I_{t+1}(X)\}$.

Proof. First consider $d = r(s-r)+1$, the dimension of the invariant ring R^G under the action of G from Remark V.5. By Main Theorem IV.8, $H^i_I(R) = 0$ for any $i > d$. Applying this again to the smaller matrix Y from Remark V.9, we see that $H^i_{I_{r-1}(Y)}(k[Y]) = 0$ if $i > (r-1)((s-1)-(r-1))+1$; in particular, $H^d_{I_{r-1}(Y)}(k[Y]) = 0$. Therefore, (V.9.2) indicates that $\left(H^d_{I_r(X)}(R)\right)_{x_{11}} = 0$. By symmetry, $H^d_{I_r(X)}(R)$ vanishes after localizing at any $x_{\alpha\beta}$, and so $H^d_{I_r(X)}(R)$ is supported only at the homogeneous maximal ideal $\mathfrak{m}$ of R.

Therefore, by Theorem IV.3, $H^d_I(R) \cong E_R(k)^{\oplus\alpha}$ for some finite integer α. Since $\operatorname{Ann}_{E_R(k)} \mathfrak{m} = k$ (see Proposition II.2.18), $\operatorname{Soc} H^d_I(R) = \operatorname{Ann}_{H^d_I(R)} \mathfrak{m}$ is a k-vector space of dimension α. By Proposition V.7, $\alpha = 1$, proving (a).

We now use induction on r, for all $s \geq r$, to prove that if $i = (r-t)(s-r)+1$ for some $0 \leq t < r$, X is an $r \times s$ matrix of indeterminates, k is a field of characteristic zero, and $R = k[X]$, then

$$\operatorname{Ass}_R\left(H^i_I(R)\right) = \{I_{t+1}(X)\}, \text{ and} \tag{V.10.1}$$

$$H^i_I(R)_{I_{t+1}(X)} \cong E_R(R/I_{t+1}(X)), \tag{V.10.2}$$

and for i not of this form, $H^i_I(R)$ vanishes. (This would prove (b) and (c).)

For the basis case, let $r = 1$. In this case, $R = k[x_1, \dots, x_s]$; if $t = 0$, then $i = r(s-r)+1 = s$. Since $I = I_1(X)$ is the homogeneous maximal ideal of R, $H^s_I(R) \cong E_R(k)$, and $H^i_I(R) = 0$ for all $i \neq s$.

Now say that for all $r_0 < r$ and all $s_0 \geq r_0$, for any $0 \leq t_0 < r_0$, if k is a field of characteristic zero and $R = k[X]$, where $X = [x_{\alpha\beta}]$ is an $r_0 \times s_0$ matrix of indeterminates, and $i = (r_0 - t_0)(s_0 - r_0) + 1$, then $\operatorname{Ass}_R\left(H^i_{I_{r_0}(X)}(R)\right) = \{I_{t_0+1}(X)\}$ and $\left(H^i_{I_{r_0}(X)}(R)\right)_{I_{t_0+1}(X)} \cong E_R\left(R/I_{t_0+1}(X)\right)$, and for all i not of this form, $H^i_{I_{r_0}(X)}(R)$ vanishes.

Take X an $r \times s$ matrix of indeterminates, $R = k[X]$, and $I = I_r(X)$. By proving (a), we have already shown (V.10.1) and (V.10.2) for $i = d = r(s-r)+1$. For $i < d$, $\mathfrak{m}$ is not an associated prime of $H^i_I(R)$ by Main Theorem IV.8 (a), so some $x_{\alpha\beta}$

must be a nonzerodivisor on $H_I^i(R)$. (If not, all products of the $x_{\alpha\beta}$, and their linear combinations, i.e., all elements of $\mathfrak{m}$, would be zero divisors.) We could renumber the indeterminates to assume that x_{11} is nonzerodivisor, but, in fact, each $x_{\alpha\beta}$ is a nonzerodivisor on $H_I^i(R)$: Consider the action of the group H described in Remark V.8. Since H is connected and $\operatorname{Ass}_R(H_I^i(R))$ is finite by Theorem IV.3 (due to the grading on R, each associated prime is contained in $\mathfrak{m}$ by Proposition II.1.8), Lemma V.2 implies that each associated prime of $H_I^i(R)$ is stable under its action. Since every indeterminate $x_{\alpha\beta}$ is in the orbit of every other indeterminate, because some $x_{\alpha\beta}$ is a nonzerodivisor (i.e., not in any associated prime), every one is a nonzerodivisor. In particular, x_{11} is a nonzerodivisor.

By the inductive hypothesis, all $H^i_{I_{r-1}(Y)}(k[Y]) = 0$ unless $0 \le t_0 < r-1$ and

$$i = ((r-1)-t_0)((s-1)-(r-1)) + 1 = (r-1-t_0)(s-r)+1,$$

or equivalently, $i = (r-t)(s-r)+1$ with $1 \le t < r$. Since each such i is less than d, x_{11} is a nonzerodivisor on $H_I^i(R)$, and (V.9.2) implies that the same vanishing conditions must hold for the $H^i_{I_r(X)}(R)$. Combining this fact with (a), we see that $H^i_{I_r(X)}(R)$ must vanish for $i < d$ unless $i = (r-t)(s-r)+1$ for some $0 \le t < r$.

Suppose that $i = (r-t)(s-r)+1$ for some $t > 0$. The inductive hypothesis tells us that $\operatorname{Ass}_{k[Y]}\left(H^i_{I_{r-1}(Y)}(k[Y])\right) = \{I_t(Y)\}$, and since S is flat over $k[Y]$ [Mat80, Theorem 12], $\operatorname{Ass}_S\left(H^i_{I_{r-1}(Y)}(k[Y]) \otimes_{k[Y]} S\right) = \{I_t(Y)S\}$. Thus, (V.9.2) implies that $\operatorname{Ass}_R\left(H^i_{I_r(X)}(R)_{x_{11}}\right)$ consists solely of $I_{t+1}(X)$, the ideal that corresponds to $I_t(Y)S$ under (V.9.1). Since x_{11} is a nonzerodivisor on $H_I^i(R)$, the associated primes of $\left(H^i_{I_r(X)}(R)\right)_{x_{11}}$ are the expansions to $R_{x_{11}}$ of the associated primes of $H^i_{I_r(X)}(R)$, and $\operatorname{Ass}_R\left(H^i_{I_r(X)}(R)\right) = \{I_{t+1}(X)\}$, proving (V.10.1).

Hochster and Eagon showed that $\operatorname{ht}_R I_t(X) = (r-t+1)(s-t+1)$ (see Theorem II.3.3), which means, in particular, that $\operatorname{ht}_{k[Y]} I_t(Y) = (r-t)(s-t) = \operatorname{ht}_R I_{t+1}(X)$.

Therefore, noting that $x_{11} \notin I_{t+1}(X)$ for any $1 \leq t < r$, we have the following sequence of isomorphisms, proving (V.10.2):

$$\left(H^i_{I_r(X)}(R)\right)_{I_{t+1}(X)} \cong \left(H^i_{I_{r-1}(Y)}(S)\right)_{I_t(Y)S} \tag{V.10.3}$$

$$\cong \left(H^i_{I_{r-1}(Y)}(k[Y])\right)_{I_t(Y)} \otimes_{k[Y]} S \tag{V.10.4}$$

$$\cong E_{k[Y]}\left(k[Y]/I_t(Y)\right) \otimes_{k[Y]} S \tag{V.10.5}$$

$$\cong E_{k[Y]_{I_t(Y)}}\left(k[Y]_{I_t(Y)}/I_t(Y)k[Y]_{I_t(Y)}\right) \otimes_{k[Y]} S \tag{V.10.6}$$

$$\cong \left(H^{\operatorname{ht} I_t(Y)}_{I_t(Y)}(k[Y])\right)_{I_t(Y)} \otimes_{k[Y]} S \tag{V.10.7}$$

$$\cong H^{\operatorname{ht} I_t(Y)}_{I_t(Y)}\left(k[Y] \otimes_{k[Y]} S\right)_{I_t(Y)} \tag{V.10.8}$$

$$\cong H^{\operatorname{ht} I_t(Y)}_{I_t(Y)}(S)_{I_t(Y)} \tag{V.10.9}$$

$$\cong H^{\operatorname{ht} I_{t+1}(X)}_{I_t(Y)S_{I_t(Y)}}\left(S_{I_t(Y)}\right) \tag{V.10.10}$$

$$\cong H^{\operatorname{ht} I_{t+1}(X)}_{I_{t+1}(X)R_{I_{t+1}(X)}}\left(R_{I_{t+1}(X)}\right) \tag{V.10.11}$$

$$\cong E_{R_{I_{t+1}(X)}}\left(R_{I_{t+1}(X)}/I_{t+1}(X)R_{I_{t+1}(X)}\right) \tag{V.10.12}$$

$$\cong E_R\left(R/I_{t+1}(X)\right). \tag{V.10.13}$$

(V.10.3) and (V.10.11) are induced by (V.9.1). We have (V.10.4) and (V.10.8) because S is flat over $k[Y]$ (see Theorem II.2.22). (V.10.5) is by the inductive hypothesis. Since R is Gorenstein, we have (V.10.7) and (V.10.12) (see Theorem II.2.21). Proposition II.2.16 provides (V.10.6) and (V.10.13), and Corollary II.2.23 provides (V.10.10).

□

CHAPTER VI

Vanishing of $H^1_{\mathfrak{m}}\left(H^i_I(R)\right)$

Throughout this chapter, suppose that k, R, $I_t(X)$, and I satisfy Hypothesis V.3, and let $\mathfrak{m}$ denote the homogeneous maximal ideal of R.

Remark VI.1 (The iterated local cohomology modules $H^0_{\mathfrak{m}}(H^i_I(R))$**.** By the Main Theorem on Minors, V.10, $H^0_{\mathfrak{m}}(H^i_I(R)) = 0$ unless $i = r(s-r)+1$, in which case, $H^0_{\mathfrak{m}}(H^i_I(R)) \cong E_R(k)$ since every element of $E_R(k)$ is killed by a power of $\mathfrak{m}$.

Other iterated local cohomology modules of particular interest are those of the form $H^1_{\mathfrak{m}}(H^i_I(R))$, since their vanishing helps us characterize $H^i_I(R)$:

Remark VI.2 (Vanishing of the iterated local cohomology modules $H^1_{\mathfrak{m}}(H^i_I(R))$**.** Suppose that $H^1_{\mathfrak{m}}(H^i_I(R)) = 0$, where $i = (r-t)(s-r)+1$ for some $0 < t < r$. By Theorem V.10, $H^i_I(R)$ injects into $E_R(R/I_{t+1}(X))$, and $I_{t+1}(X) \subsetneq \mathfrak{m}$. If we call the cokernel of this injection C, we have a short exact sequence

$$0 \to H^i_I(R) \to E_R(R/I_{t+1}(X)) \to C \to 0,$$

which gives rise to the long exact sequence in local cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0_{\mathfrak{m}}(H^i_I(R)) \arrow[r] & H^0_{\mathfrak{m}}(E_R(R/I_{t+1}(X))) \arrow[r] & H^0_{\mathfrak{m}}(C) \arrow[dll, rounded corners] \\ & H^1_{\mathfrak{m}}(H^i_I(R)) \arrow[r] & H^1_{\mathfrak{m}}(E_R(R/I_{t+1}(X))) \arrow[r] & \dots \end{tikzcd}$$

Since the $I_{t+1}(X)$ is the only associated prime of $E_R(R/I_{t+1}(X))$ and of $H_I^i(R)$, $H_{\mathfrak{m}}^0(H_I^i(R)) = H_{\mathfrak{m}}^0(E_R(R/I_{t+1}(X))) = H_{\mathfrak{m}}^1(E_R(R/I_{t+1}(X))) = 0$, which implies that $H_{\mathfrak{m}}^0(C) \cong H_{\mathfrak{m}}^1(H_I^i(R)) = 0$.

Let $M := \bigcap_{1\le\alpha\le r, 1\le\beta\le s} H_I^i(R)_{x_{\alpha\beta}}$. Since $I_{t+1}(X) \subsetneq \mathfrak{m}$, $M \subseteq H_I^i(R)_{I_{t+1}(X)}$. By the Main Theorem V.10, $H_I^i(R)_{I_{t+1}(X)} \cong E_R(R/I_{t+1}(X))$, so $M \hookrightarrow E_R(R/I_{t+1}(X))$. This induces an injection $M/H_I^i(R) \hookrightarrow E_R(R/I_{t+1}(X))/H_I^i(R) \cong C$. By definition of M, every element of $M/H_I^i(R)$ is killed by a power of $\mathfrak{m}$, so applying $H_{\mathfrak{m}}^0(-)$, we see that $M/H_I^i(R) = H_{\mathfrak{m}}^0(M/H_I^i(R)) \hookrightarrow H_{\mathfrak{m}}^0(C) = 0$. Thus, $M = H_I^i(R)$, and we have the following characterization of $H_I^i(R)$:

$$H_I^i(R) = \bigcap_{1\le\alpha\le r, 1\le\beta\le s} H_I^i(R)_{x_{\alpha\beta}}.$$

In Theorem VI.3, we will prove that the $H_{\mathfrak{m}}^1(H_I^i(R))$ vanish in a special case, so we have this characterization. We make use of the spectral sequence described in Remark II.5.1 in the case that I is as defined, $J = \mathfrak{m}$, and $M = R$; i.e.,

$$E_2^{p,q} = H_{\mathfrak{m}}^p(H_I^q(R)) \underset{p}{\Longrightarrow} E_\infty^{p,q} = H_{\mathfrak{m}}^{p+q}(R).$$

Theorem VI.3. *Assume that k, R, and I satisfy Hypothesis V.3, and that $r = 2$ and $s \ge 3$, so that I is generated by the 2×2 minors of the $2 \times s$ matrix X. Then*

$$H_{\mathfrak{m}}^0\left(H_I^{2s-3}(R)\right) \cong H_{\mathfrak{m}}^{s-1}\left(H_I^{s-1}(R)\right) \cong H_{\mathfrak{m}}^{s+1}\left(H_I^{s-1}(R)\right) \cong E_R(k),$$

and all other $H_{\mathfrak{m}}^j(H_I^i(R))$ vanish. In particular, $H_{\mathfrak{m}}^1(H_I^i(R)) = 0$ for all i.

Proof. By Main Theorem V.10, we know that the only two nonzero local cohomology modules of the form $H_I^i(R)$ are $H_I^{2s-3}(R) \cong E_R(k)$ and $H_I^{s-1}(R) \hookrightarrow E_R(R/I)$.

Since every element of $E_R(k)$ is killed by the homogeneous maximal ideal $\mathfrak{m}$, $H_{\mathfrak{m}}^0\left(H_I^{2s-3}(R)\right) \cong E_R(k)$, and because $E_R(k)$ is a zero-dimensional module, all higher

$$
\begin{array}{c|cccccccccc}
\boxed{p} & & & & & & & & & \\
 & & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
s+2 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\
s+1 & \cdots & 0 & \boxed{H^{s+1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right)} & 0 & \cdots & 0 & 0 & 0 & \cdots \\
s & \cdots & 0 & \boxed{H^{s}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right)} & 0 & \cdots & 0 & 0 & 0 & \cdots \\
s-1 & \cdots & 0 & \boxed{H^{s-1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right)} & d^{2,2s-3}_{s-1} & & 0 & 0 & 0 & \cdots \\
\vdots & & \vdots & \vdots & d^{1,2s-3}_{s-1} & & & \vdots & \vdots & \\
2 & \cdots & 0 & H^{2}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right) & d^{0,2s-3}_{s-1} & & & \boxed{0} & 0 & \cdots \\
1 & \cdots & 0 & H^{1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right) & 0 & \cdots & & \boxed{0} & 0 & \cdots \\
0 & \cdots & 0 & H^{0}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right) & 0 & \cdots & 0 & \boxed{E_R(k)} & 0 & \cdots \\
\hline
 & \cdots & s-2 & s-1 & s & \cdots & 2s-4 & 2s-3 & 2s-2 & \cdots \quad \boxed{q}
\end{array}
$$

Figure VI.3.1: $E_2^{p,q} = H^p_{\mathfrak{m}}\left(H^q_I(R)\right)$

$H^j_{\mathfrak{m}}\left(H^{2s-3}_I(R)\right)$ vanish. If the 2×2 minors of a $2 \times s$ matrix vanish, the second row of the matrix must be a multiple of the first row; thus, $\dim R/I = s+1$. The only associated prime of $H^{s-1}_I(R)$ is I, so by Theorem II.2.8 (e), $H^j_{\mathfrak{m}}\left(H^{s-1}_I(R)\right) = 0$ for $j > s+1$.

Consider the spectral sequence of iterated local cohomology modules from Remark II.5.1 with I our ideal of interest, $J = \mathfrak{m}$, and $M = R$. The spectral sequence differential, $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$, drops $r-1$ columns. The difference between the two (possibly) nonzero columns in the array $E_2^{p,q}$, $2s-3$ and $s-1$, is $s-2$, so the only possibly nonzero differentials are the $d^{p,q}_{s-1} : E^{p,q}_{s-1} \to E^{p+s-1,q-s+2}_{s-1}$. (See Figure VI.3.1.) Consequently, every $E^{p,q}_{s-1} = E^{p,q}_2$ and every $E^{p,q}_s = E^{p,q}_\infty$.

The only nonzero local cohomology module $H^i_{\mathfrak{m}}(R) \cong E_R(k)$ satisfies $i = 2s$, and the only possibly nonzero $H^p_{\mathfrak{m}}\left(H^q_I(R)\right)$ such that $p+q = 2s$ is $H^{s+1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right)$. As every spectral sequence map to and from $H^{s+1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right)$ is zero, we have that $H^{s+1}_{\mathfrak{m}}\left(H^{s-1}_I(R)\right) \cong E^{s+1,s-1}_\infty = E_R(k)$. Moreover, every other $E^{p,q}_\infty = 0$.

The only nonzero domain of a differential is $E_{s-1}^{0,2s-3} \cong E_R(k)$, so the sole (possibly) nonzero differential is $d_{s-1}^{0,2s-3} : E_{s-1}^{0,2s-3} \cong E_R(k) \to E_{s-1}^{s-1,s-1}$. After taking cohomology with respect to the differentials $d_{s-1}^{p,q}$, we must get zero at both the $(s-1, s-1)$ and $(0, 2s-3)$ spots. This means that $d_{s-1}^{0,2s-3}$ must be an isomorphism, and $H_{\mathfrak{m}}^{s-1}\left(H_I^{s-1}(R)\right) = E_{s-1}^{s-1,s-1} \cong E_R(k)$. Since all other maps are zero, and after taking cohomology with respect to $d_{s-1}^{p,q}$ we must get zero at all other spots, all remaining iterated local cohomology modules vanish.

$\square$

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