VIGRE Algebra QR Review, September 16, 2002

I am not sure what the best format for the review is: my tentative plan is to pick a "theme" for each week, distribute some problems (or variants thereof) from old QR's on that theme, and then discuss them, along with a few off-theme problems for variety. But I am very open to other suggestions.

During the course of the semester I will write up notes on the material, including definitions and theorems, but these notes will (probably) not contain full proofs of the theorems. I will distribute a first installment of these at the first meeting.

I am hoping that participants will be willing to present their solutions to problems — but this is an entirely voluntary activity — no pressure. Here are some problems for next week's session (Monday, September 16, 4:10 – 5:50 p.m., 3088). If you need to come late or leave early, that's fine — don't worry about it. The theme for this week will be group theory.

— Mel Hochster

Problems

1. Let H be a subgroup of G of index 3. Prove that either H is normal, or that H has a subgroup N of index 2 in H such that N is normal in G.

2. Say that a group G has the TFF property if there is a normal subgroup N which is torsion-free and has finite index. Prove that if G has the TFF property then so does any subgroup of G.

3. Let G be a finite group with with a normal subgroup H such that both G/H' and H are nilpotent. Prove that G is nilpotent. Here H' denotes the derived (or commutator) subgroup of H.

4. Find all finite groups G with at most 3 conjugacy classes.

5. Let G be a group with $|G| = 30$.

(i) Prove that either G has a normal Sylow 3-subgroup or a normal Sylow 5-subgroup.

(ii) Prove that G is isomorphic with a semidirect product of $\mathbb{Z}_{15}$ by $\mathbb{Z}_2$. (This is equivalent to showing that G has a normal subgroup that is cyclic of order 15 and a subgroup that is cyclic of order 2 meeting it only in the identity element.)

6. Find all groups of order 21, up to isomorphism.

7. Prove that a finite simple group of even order is generated by elements of order 2.

8. Let G be a group with center Z. Suppose that G/Z is cyclic. Prove that there exists an element x of G such that every element of G can be written in the form $x^n z$ for some integer n and some $z \in Z$. From this, prove that G is abelian.

9. Let I and J be ideals of a commutative ring with identity R such that $I + J = R$. Prove that for any two positive integers m, n, $I^m + J^n = R$. (The ideal I^m is generated by all products of m elements of I.)

10. (Not a QR problem.) Consider a deck with $2N$ cards, where N is a positive integer. Call the cards $1, 2, \dots, 2N$, so that the top card has the label "1". Consider a shuffle that first divides the deck in half, reverses the order of the cards in the top half, and then interlaces the two halves with the cards alternating so that the original top card winds up on the bottom. The new positions are then $N+1$, N, $N+2$, $N-1$, $\dots$, $2N-1$, 2, $2N$, 1. Suppose that $2N = 2^n$, where n is a positive integer. Show that $n+1$ of these shuffles restore the deck to its original order.