VIGRE Algebra QR Review, December 2, 2002

Problems new this week

These problems deal with a mixture of topics.

1. (a) Define the notions of nilpotent group and solvable group. Is a solvable group necessarily nilpotent? What about the converse?

(b) Let L be the splitting field of the monic polynomial $h(x)$, with rational coefficients, over the rational numbers, and let G be the Galois group of L over $\mathbb{Q}$. Describe the connection between the problem of characterizing when all roots of $h(x)$ are expressible by radicals and the group-theoretic properties of G. Without proof, what is the group G of smallest order for which it is not possible to express all roots of $h(x)$ by radicals?

(c) If G as in part (b) has a normal subgroup H of order $128 = 2^7$ and $[G : H] = 125 = 5^3$, are the roots of $h(x)$ necessarily expressible by radicals? Explain your answer.

2. Describe, with proof, all possible Jordan forms, over the complex numbers, for a real skew-symmetric 3×3 matrix T. Can T be a non-zero nilpotent matrix?

3. Consider the eight points in $\mathbb{R}^3$ each of whose coordinates is either 1 or -1: these are the vertices of a cube. Let G be the subgroup of the real orthogonal group of $\mathbb{R}^3$ that permutes these eight points. What is the order of G? Show that the intersection of G with the diagonal matrices is the same as the intersection of the 2-Sylow subgroups of G. Is G solvable? Describe a p-Sylow subgroup of G corresponding to each prime p dividing the order of G, and determine how many p-Sylow subgroups there are. Justify your answers.

4. Consider the quotient G of the free abelian group $\mathbb{Z}^3$ by the subgroup H spanned by the elements $(4, 3, 2)$ and $(9, 7, 5)$. Express G explicitly as a direct sum of cyclic groups.

5. Let $p > 1$ be a prime integer and let G be a finite simple group whose order is divisible by p^2. Prove that every proper subgroup of G has index at least $2p$.

6. Find the Galois group of the splitting field F of $x^3 - 5$ over the rational numbers $\mathbb{Q}$. Use the structure of this group to describe completely the finite partially ordered set of subfields of F (including F and $\mathbb{Q}$). Your description should include for each subfield K a generator of K over $\mathbb{Q}$, the value of $[K : \mathbb{Q}]$, and whether K is normal over $\mathbb{Q}$, as well as indicating the containment relations among the various subfields.

7. Let k be an algebraically closed field and let $G = GL(n, k)$ act on the set M of $n \times n$ matrices by conjugation. Let S be a finite subset of k, and let X be the subset of M consisting of those $n \times n$ matrices all of whose whose eigenvalues are in S.

(a) Prove that the action of G on M restricts to an action of G on X.

(b) Prove that the action of G on X has only finitely many orbits.

(c) Now let $n = 3$. Find a formula for the number of orbits of G on X in terms of the cardinality of S.

8. Let A be an $n \times n$ real orthogonal matrix, thought of as an $\mathbb{R}$-linear map $\mathbb{R}^n \to \mathbb{R}^n$.

(a) Consider the matrix of $\bigwedge^t A$ with respect to the basis for A consisting of vectors $e_{i_1} \wedge \cdots \wedge e_{i_t}$, where $i_1 < \cdots < i_t$. Describe the entry in the spot corresponding to the pair of vectors $e_{i_1} \wedge \cdots \wedge e_{i_t}$, $e_{j_1} \wedge \cdots \wedge e_{j_t}$. Here $i_1 < \cdots < i_t$, $j_1 < \cdots < j_t$, the first vector should be thought of as determining a row of the matrix, and the second a column. (Technically, to get an actual matrix, one needs to order the basis. But the ordering does not affect the answer to this question or the next.)

(b) Is it true that if A is orthogonal then the matrix constructed in part (a) for $\bigwedge^t A$ is orthogonal? Prove your answer.

9. Let $\mathbb{Q}[x]$ be the polynomial ring in one variable over the rational numbers $\mathbb{Q}$ and let M be the $\mathbb{Q}[x]$-module whose underlying $\mathbb{Q}$-vector space is $\mathbb{Q}^3$, and such that the action of x on an element of M is given by the $\mathbb{Q}$-linear transformation A with matrix $\begin{pmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ -1 & 0 & -2 \end{pmatrix}$.
Express M as a direct sum of cyclic modules over $\mathbb{Q}[x]$.

10. Let A and B be $n \times n$ matrices over the real numbers such that $AB = 0$.

(a) Prove that the sum of the ranks of A and B is at most n.

(b) Show by example that BA can be nonzero.

(c) Must BA be nilpotent? Prove your answer.