VIGRE Algebra QR Review, December 9, 2002

Problems new this week

These problems deal with a mixture of topics.

1. Let F be the splitting field over the rational numbers $\mathbb{Q}$ of the polynomial $x^4 - 3$.

(a) Explain why the Galois group of F over $\mathbb{Q}$ may be thought of as a subgroup G of the permutations of the set $\{\sqrt[4]{3},\ -\sqrt[4]{3},\ i\sqrt[4]{3},\ -i\sqrt[4]{3}\}$ where f is the positive real fourth root of 3, and $i^2 = -1$. Note that $i \in F$.

(b) Show that $x^4 - 3$ is irreducible over $\mathbb{Q}(i)$, and determine $[F : \mathbb{Q}]$.

(c) Prove that G is generated by the transposition $\tau = (i\sqrt[4]{3}, -i\sqrt[4]{3})$ and the 4-cycle σ corresponding to the automorphism of F that fixes i and $-i$ and sends $\sqrt[4]{3}$ to $i\sqrt[4]{3}$. Complete the description of σ as a 4-cycle $(\sqrt[4]{3},\ i\sqrt[4]{3},\ c,\ d)$, i.e., what are the explicit values of c and d? Conclude that $G \cong$ the group of symmetries of a square.

(d) How many subgroups of G have order 2, and how many cyclic subgroups have order 4? You may assume that G has two subgroups of order 4 that are not cyclic. How many subgroups does G have in all? How many fields K are there with $\mathbb{Q} \subseteq K \subseteq F$?

(e) Assume that G has a unique normal subgroup of order 2. How many of the field extensions of $\mathbb{Q}$ in F that have degree 4 are normal? How many of the field extensions of $\mathbb{Q}$ in F that have degree 2 are normal? Explain.

2. Let S be the ring of Gaussian integers, i.e., $\mathbb{Z}[i] \subseteq \mathbb{C}$, where $\mathbb{Z}$ is the integers and i is a square root of -1. You may assume that this ring is a principal ideal domain.

Let M be the rank 3 free module S^3, and let N be the submodule of M spanned over S by the elements $(1, i, 4)$ and $(2, 3 + 2i, 14)$. Explicitly express M/N as a direct sum of cyclic S-modules.

3. (a) Show that if every element of a group G has finite odd order, then for every element $g \in G$ there is an element $x \in G$ such that $x^2 = g$.

(b) Show that if G is a finite group and for every element $g \in G$ there is an element $x \in G$ such that $x^2 = g$, then G has odd order.

4. Let K be a field, let n be a positive integer, and let V be a vector space over K of dimension n. Suppose that $T : V \to V$ is K-linear. Is it true that T is nilpotent if and only if for all integers k, $1 \le k \le n$, the trace of $\bigwedge^k T : \bigwedge^k V \to \bigwedge^k V$ is zero? Prove your answer.

5. Let A be a $n \times n$ matrix over the complex numbers $\mathbb{C}$ that satisfies the equations $A^4 = A$ and $A^3 = 3A^2 - 2A$. What are the possibilities for the minimal polynomial of A? How many possibilities are there for the similarity class of A? Describe the possible Jordan forms.

6. Let K be a field and let $R = K[x]$ be the polynomial ring in one variable over K. Let M be a finitely generated torsion module over $K[x]$.

(a) Explain why M must be finite dimensional as a K-vector space.

(b) Prove that M is the cokernel of an $r \times r$ matrix A with entries in $K[x]$ for some r. (The cokernel is the quotient of $K[x]^{\oplus r}$ by the column space of the matrix.)

(c) Prove that if $M \cong R/(f_1) \oplus \cdots \oplus R/(f_k)$ is any way of writing M as a direct sum of cyclic modules, then the product of the f_j is the same, up to multiplication by a scalar in $K - \{0\}$, as $\det A$, where A is chosen as in part (b). Conclude that $\det A$ is independent, up to multiplication by a nonzero scalar, of the choice of A. (Note that both k and the size of A may vary even though M is fixed.)

7. Let $f(x)$ be a nonconstant polynomial over the rational numbers $\mathbb{Q}$ with splitting field F, and let G be the Galois group of F over $\mathbb{Q}$. Let S_n denote the symmetric group on n elements and A_n the alternating group on n elements. Suppose that $G_3 \subseteq G_2 \subseteq G_1 = G$ are subgroups of G such that G_{i+1} is normal in G_i for $i = 1, 2$. Suppose that $G_3 = A_4 \times S_4$, that $|G_2|/|G_3| = 11^7$, and that $|G_1|/|G_2| = 68$. Does this information determine whether f is solvable by radicals? If so, is it solvable by radicals? Explain your answer carefully, making explicit the statements of any major theorems that you use.

8. Determine the behavior of the rank and signature of the matrix $\begin{pmatrix} 1 & 4 & 9 \\ 4 & 17 & 38 \\ 9 & 38 & r \end{pmatrix}$ as r varies in the real numbers. For which values of r is it positive definite? For which values of r is it non-degenerate?

9. Given an integer d which is not a square, denote by $\mathbb{Z}[\sqrt{d}]$ the ring $\{a + b\sqrt{d} : a, b \in \mathbb{Z}\}$.

(a) Prove that if $d < 0$, then the group of units in $\mathbb{Z}[\sqrt{d}]$ is finite.

(b) Show that the group of units in $\mathbb{Z}[\sqrt{2}]$ is infinite.

10. Let $p \geq 3$ be a prime integer, and let G be the group of 3×3 upper triangular matrices over $\mathbb{Z}_p$ such that all diagonal entries are equal to 1.

(a) What is the order of G? On the basis of your answer, can you predict whether the center of G is trivial?

(b) Determine the possible orders of the elements of G.

(c) Is G abelian?

(d) If the center of G is non-trivial, give explicitly a non-trivial element in the center.

(e) Is G solvable? Is G nilpotent? Explain your answers.